

THE  
PHYSICAL SOCIETY  
OF  
LONDON.

---

PROCEEDINGS.

---

VOLUME XXXIII.—PART V.

AUGUST 15, 1921.

---

*Price to Non-Fellows, 6s. net, post free 6/3.*

*Annual Subscription, 30/- post free, payable in advance.*

---

*Published Bi-Monthly from December to August*

LONDON:  
FLEETWAY PRESS, LTD.,  
3-9, DANE STREET, HIGH HOLBORN, W.C.1.

---

1921.



# THE PHYSICAL SOCIETY OF LONDON.

1921-22.

## OFFICERS AND COUNCIL.

### President.

PROF. SIR W. H. BRAGG, K.B.E., M.A., F.R.S.

### Vice-Presidents.

#### WHO HAVE FILLED THE OFFICE OF PRESIDENT

SIR OLIVER J. LODGE, D.Sc., F.R.S.

SIR RICHARD GLAZEBROOK, K.C.B., D.Sc., F.R.S.

C. CHREE, Sc.D., LL.D., F.R.S.

PROF. H. L. CALLENDAR, M.A., LL.D., F.R.S.

PROF. C. H. LEES, D.Sc., F.R.S.

SIR ARTHUR SCHUSTER, Ph.D., Sc.D., F.R.S.

SIR J. J. THOMSON, O.M., D.Sc., F.R.S.

PROF. C. VERNON BOYS, F.R.S.

### Vice-Presidents.

PROF. W. ECCLES, D.Sc., F.R.S.

PROF. A. S. EDDINGTON, M.A., M.Sc., F.R.S.

THE RT. HON. LORD RAYLEIGH, F.R.S.

PROF. SIR ERNEST RUTHERFORD, D.Sc., F.R.S.

### Secretaries.

F. E. SMITH, O.B.E., F.R.S.

*"Redcot," St. James's Avenue, Hampton Hill, Middlesex.*

D. OWEN, B.A., D.Sc.

*62, Wellington Road, Enfield, Middlesex.*

### Foreign Secretary.

SIR ARTHUR SCHUSTER, Ph.D., Sc.D., F.R.S.

### Treasurer.

W. R. COOPER, M.A., B.Sc.

*82, Victoria Street, S.W.1.*

### Librarian.

PROF. A. O. RANKINE, D.Sc.

*Imperial College of Science and Technology.*

### Other Members of Council.

G. B. BRYAN, D.Sc.

C. R. DARLING, F.I.C.

PROF. C. L. FORTESCUE, O.B.E.

E. GRIFFITHS, D.Sc.

F. L. HOPWOOD, D.Sc.

E. H. RAYNER, M.A., D.Sc.

A. RUSSELL, M.A., D.Sc.

T. SMITH, B.A.

J. H. VINCENT, D.Sc., M.A.

PROF. W. B. MORTON, M.A.

XXV. *A Method for the Micro-analysis of Gases by the Use of the Pirani Pressure Gauge. By the RESEARCH STAFF OF THE GENERAL ELECTRIC CO., LTD., London. (Work conducted by N. R. CAMPBELL.)*

RECEIVED FEBRUARY 28, 1921.

(COMMUNICATED BY C. C. PATERSON, O.B.E., M.I.E.E.)

### SUMMARY.

(1) A method of analysis of gases at a pressure between 0.1 and 0.001 mm. is described, based on the characteristic vapour-pressure temperature curve of any substance.

(2) The method demands a gauge which will measure the pressures of vapours, as well as of permanent gases, over the range mentioned. The Pirani gauge is suitable for this purpose, especially if it is used in a way which is not new, but of which the advantages have not been sufficiently recognised.

### 1. PRINCIPLE OF THE METHOD.

It is well known that the relation between the vapour pressure ( $p$ ) of any substance and the temperature ( $T$ ) can be represented approximately by

$$\log p = A - \frac{B}{T - T_0}, \quad \dots \dots \dots (1)$$

which gives

$$\frac{1}{p} \frac{dp}{dT} = \frac{(\log p - A)^2}{B^2}. \quad \dots \dots \dots (2)$$

Hence  $\frac{1}{p} \frac{dp}{dT}$  becomes infinite as the vapour pressure approaches

zero at the temperature  $T_0$ , called the condensation temperature, which is characteristic of the vapour.

The least change of pressure which any gauge will measure is fixed by the value of  $dp/p$  rather than by the value of  $dp$ ; accordingly by measurements of vapour pressure it should be possible to fix with great accuracy the condensation temperature  $T_0$  and thus to identify the vapour. If the gauge contains several vapours, but no permanent gases, and its temperature is increased from a value below the  $T_0$  for the least condensible vapour, we should get a pressure-temperature curve of the form shown in Fig. 1. The temperatures at which the sudden rises in the curve occur serve to identify the vapours present, for they are the values of  $T_0$  for these vapours, while the



change in the ordinate following a sudden rise upward (such as  $0$  to  $p_1$ ,  $p_1$  to  $p_2$ , or  $p_2$  to  $p_3$ ) will serve to measure the quantity of the vapour present, if the calibration curves for the various vapours are known.

If the whole gauge were immersed in a cooling bath in order to condense the vapours, the pressure of the permanent gases and uncondensed vapours would also vary with the temperature, and the parts of the curve in Fig. 2 between the sudden rises would not be horizontal. This difficulty can be easily overcome by cooling, not the whole gauge, but only the tip of a small side tube attached to the main vessel. The vapours will condense in this side tube at the appropriate

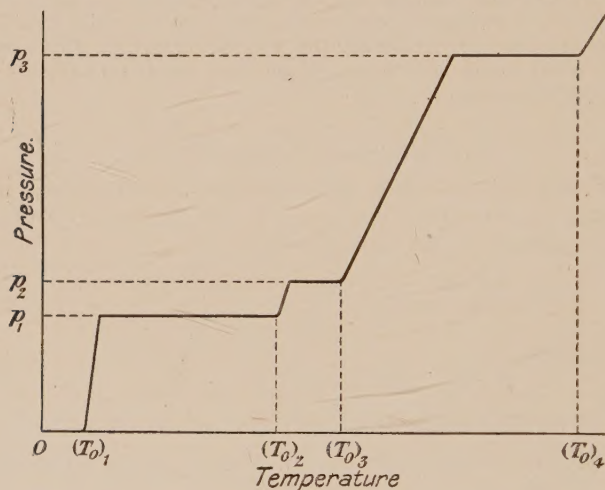


FIG. 1.

temperatures, while the readings of gauge corresponding to the permanent gases present, and to those vapours which are not condensed at the temperature of the side tube, will remain entirely unaffected. If permanent gases are present, which are not condensed at the lowest temperature attainable (which in most laboratories may be taken as that of liquid air), we shall get a curve precisely similar to Fig. 1, except that it is raised above the horizontal axis by a distance corresponding to the amount of permanent gas present.

In this statement it is assumed that the pressure of each vapour present rises from zero to the maximum, fixed by the total amount of the substance present, in the interval between

$T_0$  for that substance and  $T_0$  for the substance of next higher condensation temperature. If this assumption were not correct, the beginning of one sudden rise would be superimposed on the end of the rise preceding, and distinction between the two vapours might be difficult. But in practice at the pressures and with the substances that are likely to be present the assumption is always correct. The vapour pressure of a substance often reaches 0.1 mm. at a temperature only  $10^\circ\text{C}$ . above its characteristic  $T_0$ ; accordingly, unless either the condensation temperatures are within 10 deg. or the amount of one substance present corresponds to a pressure greater than 0.1 mm., the separate steps in the curve will be perfectly sharp. Thus, if  $\text{CO}_2$  and  $\text{H}_2\text{O}$  alone are present, the first can reach a pressure nearly atmospheric before the second begins to appear; in none of the vapours that we have desired to distinguish has there been the slightest overlapping so long as the total pressure is not more than 0.2 mm.

Little more need be said to indicate the principle of the method of micro-analysis described. If permanent gases are present, they can often be analysed by converting them into vapours, after the original vapours have been analysed, and repeating the analysis. Thus, if there is exposed to the mixture under analysis a copper wire, coated with cuprous oxide,  $\text{H}_2$ ,  $\text{CO}$ , and  $\text{O}_2$  can be identified and estimated by simply heating the wire to redness. The first two are converted into  $\text{H}_2\text{O}$  and  $\text{CO}_2$ ; the last is absorbed. The method is quite satisfactory for  $\text{H}_2$  and  $\text{O}_2$ , but  $\text{CO}$  does not seem always to be converted wholly into  $\text{CO}_2$ . But the exact methods adopted must depend largely on the nature of the problem to which the method is applied; and they will be such as are common to all methods of microanalysis.

## 2. THE PIRANI PRESSURE GAUGE.

The first requisite for the practical application of the method is a means of measuring the pressure of vapours with some accuracy up to about 0.1 mm. The Pirani gauge\* has proved very suitable. It depends on the cooling of a heated wire by the surrounding gases. In the gauge as developed by Hale† a fine wire with a high temperature-resistance

\* M. V. Pirani. Deutsch. Phys. Gesell. Verb., 8, 24, pp. 684-694, 1906.

† C. F. Hale. Amer. Electrochem. Soc. Trans., 20, pp. 243-258, 1911.



coefficient is connected in a bridge with three resistances of negligible temperature coefficient; a constant potential, sufficient to heat the manometric wire some degrees above its surroundings is applied to the bridge; the change in resistance of the wire due to its change of temperature with the pressure of the surrounding gas is measured.

This form of the gauge has been used chiefly for very low pressures (less than 0.001 mm.) for which the McLeod gauge is inconvenient. At such low pressures and, indeed, up to 0.01 mm. (with most forms of the instrument) the change of resistance is very nearly proportional to the pressure, and a straight line is obtained for the calibration curve. But at higher pressures the curve ceases to be straight, and becomes much less steep; for, as the temperature of the wire decreases, the sensitiveness of the instrument also decreases.

The shape of the calibration curve at these higher pressures cannot be represented by any simple formula; it varies in a complicated manner with the nature of the gas and the temperature of the surroundings. The calibration is purely empirical, and the instrument is not reliable in any conditions but those in which it has been calibrated. These objections can be greatly lessened by a simple change in the manner of using the gauge which was mentioned, but not fully discussed, by Pirani, and does not seem to have been widely adopted. In place of measuring the change of resistance of the wire when the potential applied to the bridge is constant, we may measure the potential which must be applied to the bridge to keep the resistance (and therefore, the temperature) of the wire constant. This potential will, of course, increase with the pressure of the gas.

In this method of using the gauge the three manganin resistances of the bridge are invariable. They are chosen so that the bridge is balanced when the manometric wire is at a convenient temperature, say, 100°C.; the wire attains such a temperature when its resistance is about one-third greater than at room temperature. A good voltmeter is connected to the terminals of the bridge, and the potential across the whole bridge varied by means of a rheostat in the battery circuit, until a balance is obtained.

The loss of heat by the manometric wire is due to radiation, to conduction along the supports, and to convection or conduction in the gas. All these losses are approximately proportional to the difference of temperature between the wire

and its surroundings ( $\theta$ ). Let the loss due to radiation and conduction along the supports (which is independent of the pressure of the gas) be  $r\theta$ , that due to the gas  $c.f(p).\theta$ , where  $p$  is the pressure. The heat supplied to the wire, since the resistance of the bridge is always the same, is proportional to  $V^2$ , where  $V$  is the potential applied to the bridge. Let it be  $\alpha V^2$ . Then

$$(r+c.f(p))\theta=\alpha V^2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If  $V_0$  is the potential required for a balance when  $p=0$ ,

$$r\theta=\alpha V_0^2. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$\frac{V^2-V_0^2}{V_0^2}=\frac{c}{r}.f(p). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now  $c/r$  is constant for considerable variations in the instrument. It does not change rapidly with the length of the wire (so long as it is so great that cooling by conduction to the supports is small); and it is surprisingly independent both of its diameter and of its material (platinum or tungsten). Moreover, so long as the changes of the temperatures of the wire and its surroundings are so small that all heat losses may be taken as proportional to  $\theta$ ,  $c/r$  does not change with those temperatures. Within the range of all the changes that are likely to occur if reasonable care is taken to make two gauges

the same, the function  $\frac{V^2-V_0^2}{V_0^2}$ , which will be written  $(V^2)$ ,

should be independent of everything except the nature and the pressure of the gas.

This independence is actually found. The manometric wire that has been employed is the tungsten filament of a 200-240-volt 40-watt lamp. The cold resistance of the lamps used varied by as much as 8 per cent., and these variations, in conjunction with changes in the room temperature (for the gauge was simply exposed to the air of the room), caused  $V_0$  to vary between 1.3 and 1.95 volts. But, in spite of these considerable variations in the instrument, if the same setting of the bridge had been used throughout and the same calibration curves used for determining  $p$  from  $(V^2)$ , the error in  $p$ , up to 0.15 mm., would not have exceeded 10 per cent. By adjusting the bridge so that the temperature of the wire was always the same, these errors could be reduced to 3 per cent.



Another great advantage of this method of using the gauge is that  $f(p)$  is approximately proportional to  $p$ , and the calibration curve approximately straight. In Fig. 2 are shown the curves for  $H_2$ , air, CO,  $CO_2$ ,  $H_2O$ . The pressure of the first four gases were measured by a McLeod gauge. For the

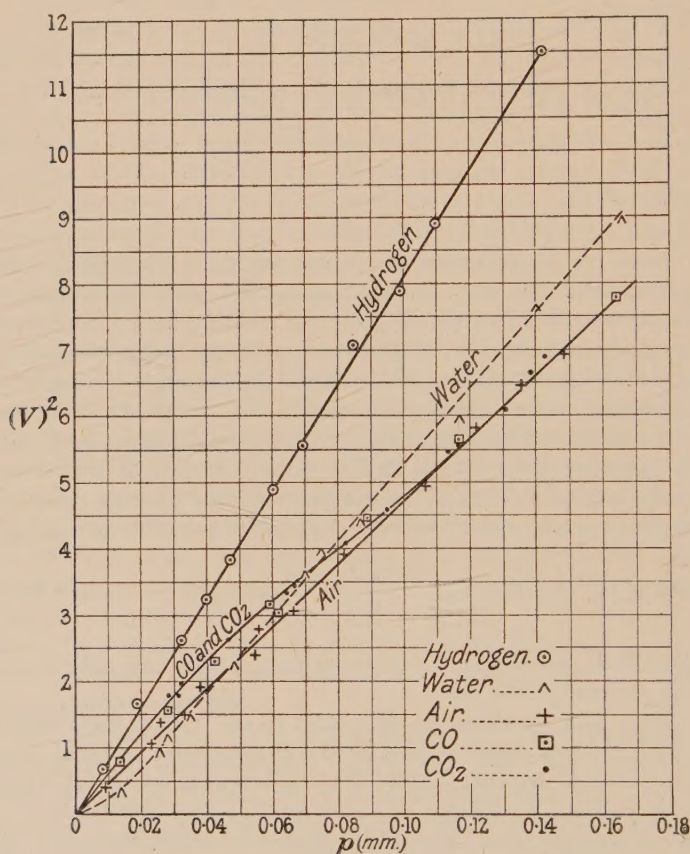


FIG. 2.

first two the curve is straight within experimental error; initial curvature appears with CO, and is more pronounced with  $CO_2$ ; but since it is known that a McLeod gauge is not really accurate for  $CO_2$ , it is possible that some of the curvature in  $CO_2$  is due to errors in measuring the pressure. The



curve for water is interesting. There is no very simple method of measuring water pressures between 0.001 and 0.1 mm., and the following device was adopted. A mixture of  $H_2$  and  $O_2$  was introduced into the vessel at a pressure measured by the McLeod; it was then burnt by a hot platinum wire, and the water formed frozen in a side tube. The pressure of the residual gas was measured (for it was difficult to secure that the gases were present in exactly the right proportion) and from the difference the amount of water formed calculated. The residual gas was pumped away, the side tube warmed, and the water liberated as vapour. The estimate of the pressure of the water made in this way assumes that none is condensed on the walls of the vessel; and this assumption is known to be untrue. The condensation of water doubtless accounts for the deviation of the curve from a straight line at low pressures, and by extrapolating the line at higher pressures to the axis, an estimate of the limiting amount of vapour condensed is obtained. The straight line probably represents the relation between  $p$  and  $(V^2)$  for the water actually present as vapour.

It will be observed that the curves for the gases which do not contain hydrogen lie very near together. For the rough measurements for which the gauge was used they may be taken as coincident. For many purposes we think that the Pirani gauge used in the manner described would have considerable advantages over the McLeod. It may, therefore, be pointed out that it has not the advantage of a greater working speed; for a period of a minute or so must be allowed for the gauge to reach thermal equilibrium.

### 3. PRACTICAL METHODS.

A few practical suggestions may be offered. If a qualitative analysis of the gases is being made, the best and most convenient way to identify the vapours is not to use any thermometer of an ordinary type but to compare the  $T_0$ 's of the unknown mixture with the  $T_0$ 's of known vapours. A series of gauges may be made up containing between them all the vapours likely to be present; any gauge of this series may, of course, contain more than one vapour so long as no interaction between constituents can occur. The side tubes from the series of known vapours and the side tube from the gauge under analysis are then immersed in a little mercury in a small Dewar flask, and liquid air poured in until the whole

is cooled to the temperature of that liquid. The conductivity of the mercury is sufficient to maintain all the side tubes at the same temperature. The mercury is then warmed by blowing air at a suitable rate on its surface, and the unknown gauge watched until a sudden change in ( $V^2$ ) occurs. The air stream is then stopped temporarily and the known gauges examined to discover in which of them ( $V^2$ ) has changed. If several have changed from their liquid air values, the constituent in the unknown gauge is, of course, that one in these known gauges which has the highest  $T_0$ .

This method is extraordinarily sensitive and certain. One per cent. of water in  $\text{CO}_2$  may be detected—an unfavourable case because the main constituent evaporates before the impurity; 0.0005 mm. of  $\text{CO}_2$  may be detected in an unlimited quantity of vapours with higher  $T_0$ , or in 0.05 mm. of permanent gases. On the other hand, mercury vapour is not easy to detect. Apparently its cooling effect is very small. But it is usually known without analysis whether mercury is present, and, if so, how much.

If the constituents are known, the estimate of their amounts can be made very rapidly. A series of baths are prepared at temperatures which lie between the  $T_0$ 's for the constituents; the side tube is then placed in the baths in succession, and the change in ( $V^2$ ) in passing from one bath to another gives the amount of the vapour with a  $T_0$  between the temperatures of the two baths.

Thus, in order to analyse a mixture containing (1) phosphorus vapour, (2) water, (3) a vapour not chemically identified from "vacuum" oil, (4)  $\text{CO}_2$ , (5)  $\text{H}_2$ , (6)  $\text{CO}$ , the following baths will serve in conjunction with the oxidised copper wire:—

Freezing mercury ( $-39^\circ$ ) condenses phosphorus but not water.

Freezing acetone ( $-95^\circ$ ) condenses water but not oil vapour.

Freezing ethyl alcohol ( $-117^\circ$ ) condenses oil vapour but not  $\text{CO}_2$ .

Liquid air ( $-183^\circ$ ) condenses  $\text{CO}_2$  but not permanent gases.

For some purposes an interpolation between the first two baths might be necessary. Then it is convenient to know



that freezing chloroform ( $-65^\circ$ ) is just above and solid  $\text{CO}_2$  ( $-78^\circ$ ) just below  $T_0$  for water. Greater difficulty arises in interpolating between the last two baths in the table, but, so far, we have not found need for such interpolation. In order to prepare the baths a few drops of the liquid are placed in a minute Dewar flask and liquid air poured in (very gently with the lighter liquids) until the solid separates.

After the values of  $V$  have been measured with the side tube immersed in the various baths, the side tube is opened and sealed to a pump, and  $V_0$  measured when the gauge is completely exhausted. If accuracy is required,  $V_0$  must be measured while the gauge is still on the pump, for in sealing it off, a small quantity of water vapour is always liberated from the melted glass. It is desirable for accuracy that the whole process should be completed as quickly as possible, in order that the room temperature may not change. It can be completed quite easily with practice in half an hour, and with reasonable care the variations during that time are not likely to be important. The most important source of error lies in the assumption, necessary unless very elaborate calculations are to be made, that the calibration curves are all straight and that the value of  $(V^2)$  for a mixture is the sum of those for its constituents. The truth of these two assumptions is probably closely connected.

Finally, we may give details of the analysis of a mixture containing  $\text{H}_2$ ,  $\text{CO}$ ,  $\text{H}_2\text{O}$ ,  $\text{CO}_2$ .

	$V$ .	$V^2$ .	$(V^2)$ .	Diff.
Side-tube at temp. of room.....	5.09	25.91	6.87	> 3.75
Freezing acetone .....	3.68	13.54	3.12	
Liquid air .....	3.24	10.54	2.20	> 0.92

After heating copper oxide—

Room .....	5.00	25.00	6.60	> 4.83
Freezing acetone .....	3.02	9.12	1.77	
Liquid air .....	1.82 <sub>2</sub>	3.31 <sub>9</sub>	0.00 <sub>8</sub>	> 1.76

Gauge exhausted on pump to  $< 0.00001$  mm.

$$1.81_5 = V_0.$$

From Fig. 2,  $p = \alpha(V^2)$ , where

$$\begin{aligned}\alpha &= 0.0121 \text{ for } \text{H}_2, \\ &= 0.0174 \text{ for } \text{H}_2\text{O}, \\ &= 0.0211 \text{ for } \text{CO} \text{ and } \text{CO}_2.\end{aligned}$$

$$\begin{aligned}
 \therefore \text{H}_2\text{O present originally} &= 3.75 \times 0.0174 = \underline{0.065_2} \text{ (0.061)} \\
 \text{CO}_2 \text{ present originally} &= 0.92 \times 0.0211 = \underline{0.019_5} \text{ (0.018)} \\
 \text{H}_2\text{O produced by burning} &= \\
 \text{H}_2 \text{ present originally} &= (4.83 - 3.75) \times 0.0174 = \underline{0.018_8} \text{ (0.020)} \\
 \text{CO}_2 \text{ produced by burning} &= \\
 \text{CO present originally} &= (1.76 - 0.92) \times 0.0211 = \underline{0.017_6} \text{ (0.020)} \\
 \text{Total.....} &0.121_1
 \end{aligned}$$

The figures in brackets are the pressures of the gas estimated when the gauge was filled for this test.

Observe that the value of ( $V^2$ ) for the permanent gases originally present should be

$$\frac{0.0188}{0.0121} + \frac{0.0176}{0.0211} = 1.55 + 0.84 = \underline{2.39.}$$

The value found was 2.20.

#### DISCUSSION.

Dr. J. S. G. THOMAS asked if the method would discriminate between the higher hydrocarbons in coal gas, or between ethane and methane. It was assumed that the heat lost by conduction at the ends of the supports was constant. He did not think this was justified, as it had been shown that at low pressures this loss depended on the pressure of the gas. Would the method discriminate between dry and wet gas at the same total pressure?

Dr. D. OWEN expressed his admiration of the method, which appeared to be entirely new. What pressures had the author gone down to in these experiments? In addition to the Pirani gauge the McLeod gauge is referred to, but no others. Had Dr. Campbell any experience of other forms of low-pressure gauge and their properties?

Dr. CAMPBELL, in reply, said it was easy to analyse any mixture of components which could be frozen out by liquid air. They had analysed the vapours from the oil used in their pumps, and he thought the higher hydrocarbons in coal gas would present no difficulty. He was afraid neither ethane nor methane would freeze out, and so could not be treated by this method. As regards the validity of the assumptions of the heat loss, for accurate work every gauge is separately calibrated, and no assumptions are made. The instrument could be used to discriminate between wet and dry gas, provided the total pressure is under 0.2 mm. He did not know how low pressures the Pirani gauge would do for, but for very low pressures the Ionisation gauge would be more accurate.



XXVI. *On the Reflection of the X-Ray Spectrum of Palladium from Fluorspar.* By H. PEALING, M.Sc.

RECEIVED MARCH 25, 1921.

ABSTRACT.

An examination of the odd order spectra reflected from the 100 plane of fluorspar, using palladium X-rays, has been made with a view to test the Lewis-Langmuir theory of the motion of valency electrons in compounds. Evidence, in partial confirmation of the theory, has been obtained.

THE reflection of X-rays by fluorspar has been investigated by Sir W. H. Bragg and Prof. W. L. Bragg, and its space lattice mapped out. The fluorine atoms are arranged in a cube lattice and the calcium atoms in a face-centred cube lattice. Each fluorine atom is surrounded by four calcium atoms and each calcium atom by eight fluorine atoms.

Calcium has atomic number 20 and fluorine 9. When all the calcium and fluorine atoms are acting so as to give a strong spectrum they will contribute to the amplitude of that spectrum in the ratio of 20 to  $2 \times 9$ , the total amplitude being proportional to 38, the arrangement of electrons within the atom being supposed, for the moment, to have no special effect. When we get a weak spectrum the calcium atoms are in opposition to the fluorine atoms, and if the atomic number law is true the amplitude of such a weak spectrum should be  $20-18$  or  $2$ . A comparison of the intensities of strong and weak spectra, account being taken of the obliquity or order of the spectra, should give us the ratio  $38^2/2^2$ , or  $361/1$ . That is, the weak spectra should be exceedingly small. The Braggs found that the first spectrum in the 1, 0, 0, plane of fluorspar was absent. The order of accuracy of the experiments could not establish whether there was a weak spectrum or none at all. This investigation was an attempt to find fresh information about the weak spectra reflected by fluorspar. It was thought that the experimental evidence of the size of these spectra would throw light on the Lewis-Langmuir theory of the structure of the atom. This theory postulates that the electrons in an atom occupy relatively fixed positions with regard to the nucleus, and are arranged in a series of shells with the nucleus as centre. The first complete shell contains two electrons, the second eight, the third eight, the fourth eighteen and the fifth eighteen.

The atoms of helium, neon, argon, krypton, and xenon contain complete shell systems, and other elements incomplete shell systems. The chemical properties of the elements depend on the number of electrons over or below a complete shell system, and in combining with other elements the tendency always is to form complete shell systems.

An electro-positive element has rather more electrons than the nearest complete shell system, while in an electro-negative element the electrons are not quite numerous enough to form a complete shell system. If an electro-positive element combines with an electro-negative element then the electrons over and above the complete shell system, the valency electrons as they are sometimes called, can pass over to the electro-negative element to enable it to form a complete and therefore stable shell system. A calcium atom having two valency electrons will combine with two fluorine atoms and the resulting combination will give a calcium atom with an argon shell system, and two fluorine atoms with neon shells, the atomic number of argon being two less than that of calcium and the atomic number of neon being one more than that of fluorine.

In what way will the calcium atom with the argon shell of electrons scatter X-rays? If it scatters so that the amplitude of the wavelet depends only on the number of electrons in the atom, then it will behave as an atom of argon and the fluorine atom should behave as an atom of neon. Consequently when the calcium and fluorine atoms are acting in opposition when giving a spectrum, we should get an amplitude proportional to  $\{(20-2)-2(9+1)\}$ , or one proportional to 2. It was considered possible to detect whether a small spectrum of this magnitude was present or not. In magnitude it is  $1/361$  of the intensity of a full spectrum. We shall give the data later, which we consider establishes the fact that a small first 1, 0, 0, spectrum is present, but only about one-third of the value expected. It does not seem possible to reconcile this fact with the scattering law stated above, without qualifications. It seems possible, however, to modify slightly the Lewis-Langmuir theory so as to explain it. Although the argon and neon shells may be formed, it does not follow that the shell systems will be identical with those characteristic of those elements. They may be the argon and neon shells slightly distorted, that is to say, with the electrons slightly out of position. It does not seem reasonable



to suppose that the nuclear charges of the calcium and fluorine atoms change. If they do not change we must suppose that the unbalanced positive charges, on the nucleus of the calcium atom, produced by the absorption of the valency electrons into the fluorine atoms, must displace the remaining electrons in the rings, slightly, from their true positions.

In the 1, 0, 0, plane of flourspar the calcium and fluorine atoms lie on alternate planes which are all equidistant. If a distortion takes place, then the phase of the disturbance contributed by the fluorine atoms in any one plane, may not be just the mean of the disturbances from the two calcium planes, on either side of the fluorine plane. Further, the amplitude of the disturbance contributed by each atom will be altered to a slight extent.

In the experimental arrangements just the same method and apparatus was used as in the previous work of Sir W. H. Bragg and of his son, Prof. W. L. Bragg, an excellent account of which is given in their book "X-rays and Crystal Structure." The specimen of flourspar used was a colourless variety, and the 1, 0, 0, face had been carefully worked. The writer attempted to find the  $\alpha$  lines of the 1st, 3rd, 5th and 7th spectra reflected from that plane. In the first place only the 3rd was observed. On taking greater care the 1st was afterwards observed. The third spectrum is quite easy to observe. Two separate sets of observations taken with the bulb and electroscope under entirely different conditions gave the magnitude of the 3rd spectrum as 4.1 per cent., and 3.6 per cent. of the 2nd spectrum. We may take the value 4 per cent. as very nearly right. Other sets not taken with the same closeness of observation gave similar values. There is a slight uncertainty with regard to the position of the line. The angle of reflection was a little less than we should expect.

In observing the first order spectrum, care was taken to eliminate variations in the bulb and electroscope while a set of observations was being taken. After preliminary observations had indicated the line, the following procedure was adopted. In order to test any variations in the bulb or electroscope while the experiments were in progress, observations of the large 2nd order spectrum were made before and after a set of observations on the 1st order spectrum. The writer was unable to avoid a slight hardening of the bulb as the experiment proceeded, but this only caused a 5 per cent. reduction in the line spectrum. As the hardening of the bulb

was progressive, in the writer's experiments, four sets of observations were taken. The 1st and 3rd sets were made with gradually increasing glancing angle, and the 2nd and 4th with decreasing angles. The average of one set of readings is shown in graph form in Fig. 1. The ordinates represent

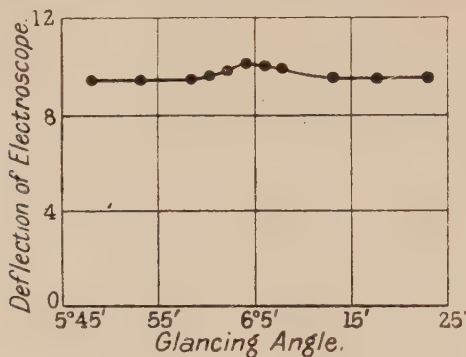


FIG. 1.— $\alpha$  LINE, 1ST ORDER.

the deflections of the electroscope, and therefore very approximately the intensities of the reflected rays, and the abscissæ the glancing angles. The time for each deflection was 25 seconds. The bulb used was of the gas type. It will be seen

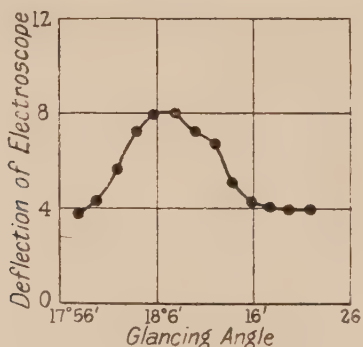


FIG. 2.— $\alpha$  LINE, 3RD ORDER.

that the curve is unusually flat, except for the glancing angles, where we would expect an  $\alpha$  line if such existed. Several other sets gave similar results. The intensity is very small indeed, about  $1/800$  of a full spectrum, and the difference between this and the value expected—viz., 1.361—is somewhat greater



than the order of experimental error. The divergence in the case of the third order spectrum was far greater. The form of the  $\alpha$  line is shown in Fig. 2. The scale is not far different from that of Fig. 1. The value observed is about thirty times that expected.

The distortion theory the writer has put forward would

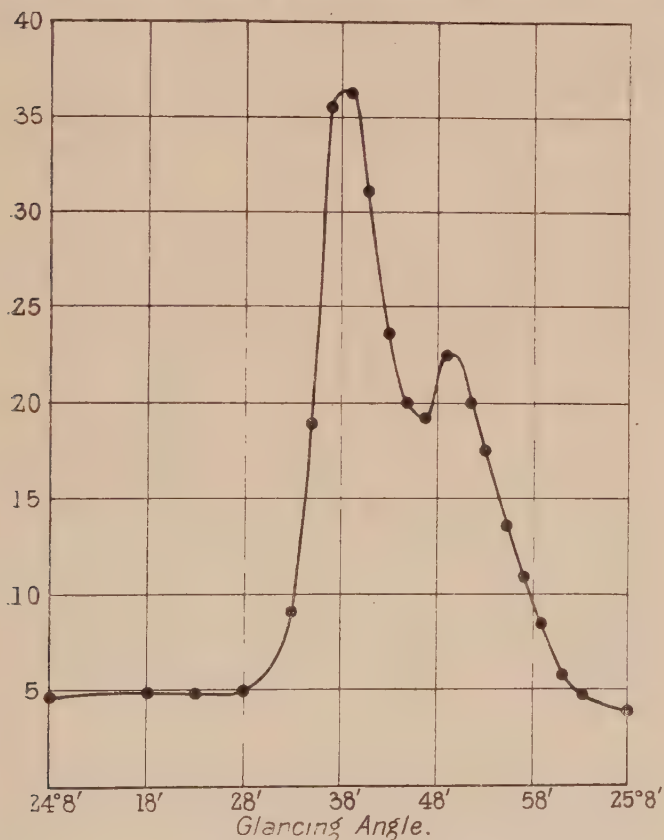


FIG. 3.— $\alpha$  LINES, 4TH ORDER.

easily account for the magnitude of the 1st order observed, but it could only be made to account for the size of the 3rd order by supposing that the ratio of reflecting powers of the two sets of atom planes altered according to the order of the spectrum; in other words, that it depended on the obliquity of the rays. This would be so, if the unit of reflection was that from each

shell electron, and not that from the atom as a whole. In this case the contribution of each electron in any one atom will vary according to the position of the electron with regard to the nucleus and the wavelets from each electron will not be in the same phase. The total from each atom will be the vector sum of the wavelets contributed by each electron from its shell system. A study of the even order spectra in the same plane affords additional confirmation of these views. The 4th, 6th and 8th spectra were compared with the 2nd, and they were found to have intensities of 40, 11 and 3 per cent. respectively of the 2nd, whereas the percentages to be

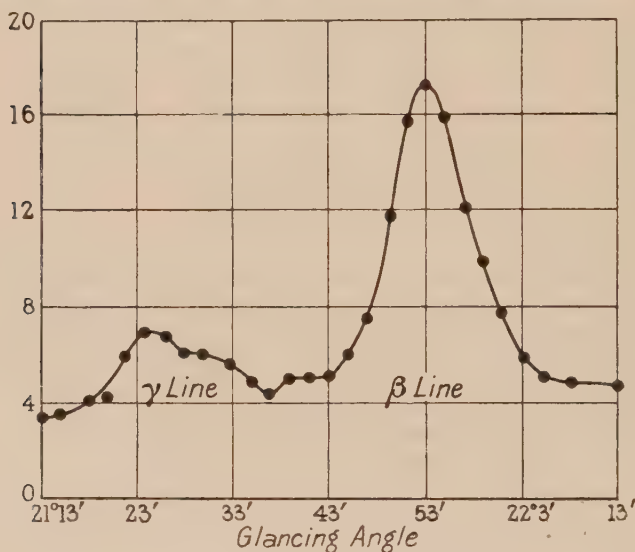


FIG. 4.— $\beta$  AND  $\gamma$  LINES, 4TH ORDER.

expected were 25, 11 and 6 per cent. respectively. The fourth order is seen to be unusually large. It is shown in detail in Figs. 3 and 4. Fig. 3 shows the two  $\alpha$  lines, and Fig. 4 the  $\alpha$  and  $\beta$  lines. The time taken for each reading was 20 seconds. It will be seen that lines usually regarded as faint are quite easy to observe by this method in high order spectra.

We have tried above to explain the large third order spectrum by supposing that the electron is the source of the diffracted wavelets. For an obliquity corresponding to the 3rd order we must suppose that the wavelets from the electrons

of one of the atoms—say, calcium—are more than usually in agreement so far as phase is concerned. For a slightly greater obliquity, the other atom—say, fluorine—exhibits the same sort of behaviour, and consequently we must expect an augmented effect. The third spectrum on this view is due to the increased reflecting power of the one set of atoms, and the large fourth order due to the increased reflecting power of the other atoms.

Sir W. H. Bragg suggested this investigation to me, provided me with the spectrometer and crystals, and gave me much kind help during the course of the work.

#### DISCUSSION.

Prof. BRAGG said that the relatively high magnitude of the third order spectrum was most curious. There was evidently something diffracting strongly about a third of the way between the calcium and fluorine atoms: It was possibly only a coincidence, but this was where the junction of these atoms occurred.

Prof. RANKINE asked what was supposed to be the source of the diffraction, the nucleus or the electrons distributed round it. If the latter, what was the justification for assuming the diffraction to originate from the planes containing the nuclei?

Prof. BRAGG said the origin of the diffraction was what they were all trying to settle. Possibly the Paper he was just about to read would answer Prof. Rankine's question.

Dr. HOPWOOD was not clear as to what the Paper proved. Assuming the reflecting power of the calcium and fluorine atoms to be proportional to the atomic numbers the same magnitude for the faint spectra was apparently to be expected with or without the transference of electrons of the Lewis-Langmuir theory. This being so, he did not see that the result was evidence one way or the other.

Mr. PEALING said the discrepancy of the larger third order spectrum was more easily accounted for on the basis of the Lewis-Langmuir theory.

Dr. HOPWOOD thought this might be due to a single distortion rather than to a transfer of electrons from the calcium to the fluorine.



XXVII. *The Intensity of X-Ray Reflection by Diamond.*  
By SIR W. H. BRAGG, K.B.E., F.R.S.

RECEIVED MAY, 13, 1921.

ABSTRACT.

The Paper describes an investigation of the relative intensities of the reflections of monochromatic X-rays by the various crystallographic planes of diamond. The special difficulties due to the small size of available crystals and the modification of the usual method required to obviate them are discussed. An interesting feature of the results is that they lie very closely on smooth curves, which indicates that if the outer electrons of the carbon atom lie at any considerable distance from the centre they must be in motion over a wide range, or for some other reason must contribute little to the reflection.

It is shown that the properties of the carbon atom in diamond are based on a tetrahedral and not a spherical form. The tetrahedra point away from any (111) plane in the case of half the atoms, and towards it in the case of the other half. Consecutive 111 planes are not exactly of the same nature and consequently some slight second order reflection from the tetrahedral plane might be expected. This effect, though slight, has been found.

A KNOWLEDGE of the relative intensities of the reflections of monochromatic X-rays by the various planes of diamond might be expected to give useful information for several reasons.

The most important is the general reason that such relative intensities for any crystal depend on the arrangement of the atoms round each point on the lattice and on the arrangement of the electrons in each atom, and are markedly characteristic of each crystal. Nevertheless, because of insufficient knowledge, it is not easy to interpret the results.

Consequently, a study of these intensities in the case of diamond might be particularly useful, because there is but one kind of atom and because the structure is simple and exactly known. Interpretation might be easier than in the case of more complicated crystals.

There is, however, one difficulty which is not met with in the case of rock-salt or other crystals which can show faces of large area, either natural or prepared. When a fine pencil of X-rays is allowed to fall on a large enough face the dimensions of the crystal need not enter into the calculations; because the face may be so large that the pencil cannot miss it, and the depth of the crystal below the face may be so great that the rays are wholly absorbed.

This cannot be realised in the case of diamond, and a different plan must be followed. Instead of making the pencil of rays small, and using a large crystal, it is necessary to make the pencil so broad that the diamond is entirely bathed in it. All the slits of the spectrometer are opened wide, and the dimensions of the reflected pencil are determined by the size of the crystal.

The diamond is mounted on a stand somewhat similar to that of a goniometer, and its position is so adjusted by means of the X-ray reflections that any convenient zonal axis coincides with the axis of the spectrometer. It is then easy to revolve the crystal from one position to another, and to test the intensity of reflection by any of the planes belonging to the zone.

The zonal axis most used was that which is determined by the intersection of (110) and (001); all the important planes pass through it. In measuring the intensity for any particular plane, the ionisation chamber was first set at the proper angle, and the crystal was then turned at a uniform and definite rate through the small range of angle for which it reflects. The intensity was measured by the calibrated movement of the electroscope leaf during the sweep. Allowance was made for the small amount of general radiation which entered the chamber during the time of observation. The method has been described in the "Philosophical Magazine," May, 1914.

The results shown in Table I. were obtained with a diamond weighing 9.8 milligrams. When the current in the Coolidge bulb was 1 milliampere and the potential about 40,000 volts, and when the electroscope was adjusted to a moderate sensitiveness, such as 20 divisions to the volt, the reflection from the 111 plane caused the leaf to move at a rate of about 5 divisions in a second. The divisions were such as could be readily divided into 10 by eye. The bulb had a rhodium anticathode.

The diamond\* was chosen of a form as nearly as possible spherical; it showed no well-marked faces. This was done because it was expected that the reflected intensities would show the influence of absorption within the crystal, as actually proved to be the case. If the diamond had been very irregular

\* This and other stones were most kindly given to me by Mr. Alphonse Abrahams.

in shape, the intensities corresponding to the different planes would have been affected thereby in different degrees, and it would have been very difficult to make the correct allowances. This difficulty was largely avoided by choosing a diamond nearly spherical in form, and by measuring the intensities of all the planes belonging to three out of the six different zonal axes of the type already mentioned. The intensities agreed among themselves to about 10 per cent. The figures in Table I. are averages, and are probably correct to 5 per cent. They are given exactly as they were found, without allowance for polarisation, temperature or other influence.

TABLE I.

Indices of plane.	Cosec $\theta$ .	Intensity.
111	6.72	200 (standard).
110	4.10	145
311	3.51	84
222	3.36	4
100	2.31	97
331	2.67	62
211	2.38	81
333 } 511 }	2.24	45
220	2.05	62
310	1.83	52*
533	1.76	29
444	1.68	38
711 } 551 }	1.63	21.5
321	1.56	36*
553	1.51	19
200	1.45	28
733	1.42	16.5
411 } 330 }	1.37	26
555	1.34	11.5

The results are plotted in Fig. 1, the ordinates being the intensities and the abscissæ the cosecants of the angles of reflection.

The curves show certain well-marked features. All the planes which have any even numbers in their indices lie on one smooth line; the rest lie on another. This is obviously due to the fact that the spacings of planes of the first kind are all equal to each other. The spacings of the second kind are alternately as 1 to 3. This is most easily realised in the case of the (111) or tetrahedral plane, and is a consequence of the fact that every atom is at the centre of gravity of its four nearest neighbours.

\* Only single instances of these planes were examined.



In the case of the first and third order reflections from any plane whose indices are all odd, the reflections from alternate sheets of atoms are in quadrature with each other. In the case of the fourth order reflection all sheets act together. For these reasons the (444) point lies on the

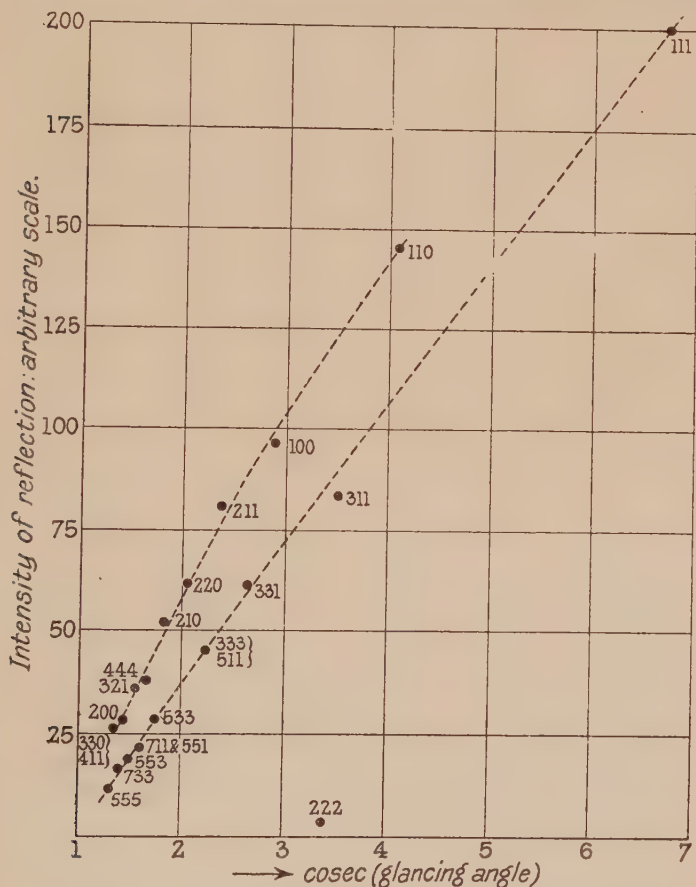


FIG. 1.—RELATIVE INTENSITIES OF REFLECTION FROM VARIOUS PLANES OF DIAMOND.

upper curve in the figure, but the points representing the intensities of (111) and (333) lie on the lower.

The influence of the arrangement which is characteristic of diamond structure and which causes alternate sheets of a set to act at one time in unison with each other and at

another time in quadrature should cause the ordinates of the upper curve to be twice those of the lower. For if we start with a set of equal and equidistant sheets, and then take half the atoms out of each sheet to form a new set of sheets which divide the spacings of the old set in the ratio 1 : 3, the intensity of the reflection of the  $n$ th order is diminished in the ratio

$$1 : \{(1 + \cos n\pi/2)^2 + \sin^2 n\pi/2\} / 4.$$

If  $n=1$  or  $= 3$ , the ratio is 2 : 1.

This is far from being the case, particularly on the right of the diagram. The fact is that absorption of the rays has to be taken into account, and in particular, the special absorption which occurs when the crystal is so arranged as to be reflecting.

When a crystal is made to revolve through its range of reflection as in these experiments the total intensity of the rays reflected by a small element of the crystal is proportional to the intensity of the rays incident on the element and to the volume of the element. The pencil of rays that has passed through the element has lost energy on account of the reflection that has taken place. A similar loss or absorption of energy of the original pencil takes place all along its path, while reflection is occurring. The ordinary absorption co-efficient  $\rho$  is supplemented by a special absorption co-efficient which may for present purposes be assumed to be proportional to the amount of reflection; and to be capable of addition to the former. This co-efficient  $r$  will be different for different planes, and for any one plane it will vary with the glancing angle, having a sensible value for only a very small range on either side of the true angle of reflection.

The primary pencil has to traverse a certain distance in the crystal before it reaches any element of volume  $dv$ ; and the reflected pencil has to traverse a further distance before it emerges. Along both these paths the absorption co-efficient is equal to  $\rho + r$ . If  $d$  is the sum of the two paths, the contribution of the element to the reflection intensity is proportional, for any given value of the glancing angle, to

$$rdve^{-(r+\rho)d}.$$

Secondary and further reflections are neglected.

The intensity of the whole reflection is obtained by integrating this expression over the volume of the crystal. The result depends on the shape of the crystal as well as on its

volume, and also on the values of  $r$  and  $\rho$ . For our present purpose, a very approximate estimate is enough. It is sufficient to suppose the crystal to be so small that only first powers of  $(r+\rho)d$  need be retained.

If  $(r+\rho)d$  is neglected altogether the intensity of reflection is proportional to  $rV$  where  $V$  is the volume of the crystal. If first powers are retained the integration of  $rdv(1-\overline{r+\rho.d})$  will yield an expression  $rV(1-\overline{k.r+\rho})$  where  $k$  is a linear quantity depending on the form of the crystal.

If, then, a point on the upper curve of Fig. 1 is compared with a point on the lower for which the abscissa is the same, the ratio of the ordinates ought to be less than two to one. The ratio would be exactly two to one if the crystal was extremely small, but, taking in first powers of  $(r+\rho)d$  it is now, for the particular glancing angle in question, of the form

$$2. \frac{1-k(2r+\rho)}{1-k(r+\rho)},$$

which is always less than 2 and diminishes as  $r$  increases.

When, as in these experiments, the crystal is made to revolve through an angular range which includes the true angle of reflection, the numerator and denominator of this fraction require separate integration over that range; but the final ratio is still less than 2, and the more so the stronger the reflection.

If the diamonds are so large that it is not enough to limit the calculation to first powers of  $(r+\rho)d$ , the general conclusion is not affected. So also, if diamonds of different weight are compared with each other the larger crystals should show the effect of absorption more than the smaller, since  $k(r+\rho)$  increases with the dimensions of the diamond.

The comparative intensities for a few diamonds of different weight are shown below :—

TABLE II.						
Weight in mgm.		(111)		(333)		(444)
61.4	...	100	...	33.5	...	26
9.8	...	45	...	10.4	...	8.5
4.55	...	34	...	7.0	...	5.4

The largest stone weighs more than thirteen times the smallest, but the (111) reflection is only three times larger, and even the (444) reflection, for which  $r$  is smaller, is only five times larger than the corresponding reflection of the small stone. In the case of a large diamond of a roughly triangular



form, and of more than 2 mm. thickness, the relative intensities for the three planes of Table II. were 100 to 45 to 39. This stone was kindly lent to me by Sir George Beilby.

Perhaps the most remarkable feature of the results represented in the diagrams is that such smooth curves can be drawn through the various points. If the outer electrons of the carbon atom lie at any considerable distance from the centre they must be in motion over a wide range,\* or for some other reason they must contribute little to the reflection. For if not, their effects on different planes would vary greatly. If they all lie close to the centre such large motions need not be assumed. The question may be settled by an absolute measure of the intensity of diamond reflections, but the experiments are not yet complete.

Another point of interest is the existence of a small 222 reflection. This has been looked for previously but without success. The structure of the diamond cannot be explained on the hypothesis that the field of force round the carbon atom is the same in all directions: or in other words, that the force between two atoms can be expressed simply by a function of the distance between their centres. If this were so the spheres, which would then represent the carbon atoms appropriately, would adopt the close-packed arrangement. Each atom would surround itself with twelve others, all similarly placed. As a matter of fact, each atom is surrounded by four neighbours only, and the structure is so hollow that it is possible to add to the atoms in a given space others in number equal to those already there. The arrangement then becomes that of the centred cube, and each atom has eight neighbours instead of four.

It is necessary, therefore, to suppose that the attachment of one atom to the next is due to some directed property, and that the carbon atom has four such special directions: as indeed the tetra-valency of the atom might suggest. In that case the properties of the atom in diamond are based upon a tetrahedral not a spherical form. The tetrahedra point away from any (111) plane in the case of half the atoms in the diamond and towards it in the case of the other half. Consecutive 111 sheets are not exactly of the same nature; and it might reasonably be expected that they would not entirely destroy

\* See Papers by Coster, Proc. Roy. Acad. Sci., Amsterdam, Oct., 1919; and by Kolkmeijer, *ibid.* Jan. 1920.

each other's effects in the second order reflection from the tetrahedral plane. It is this effect which is now found to be quite distinct, though small.

## DISCUSSION.

Dr. ANDRADE asked if surface tension effects entered into the problem.

Dr. RAYNER asked if there was any chance of obtaining the arrangement of carbon atoms in the benzene ring.

Prof. BRAGG explained that surface tension effects did not come in. In reply to Dr. Rayner, he exhibited a model of the benzene group.

## EXHIBIT OF PHOTOGRAPHS BY M. LE DUC DE BROGLIE.

Prof. BRAGG exhibited and explained photographs by M. le Duc de Broglie of the "Magnetic Spectra" obtained when the  $\beta$ -rays produced when X-rays strike copper are deflected on to a photographic plate by a magnetic field.

XXVIII. *Capacity and Eddy Current Effects in Inductometers.*  
*By S. BUTTERWORTH, M.Sc. (The National Physical*  
*Laboratory.)*

RECEIVED MARCH 23, 1921.

ABSTRACT.

1. A theoretical investigation has been made upon the effects of self and intercapacities and of eddy-current losses in mutual inductances for which the coils have a common point. The following are the main conclusions arrived at.

(a) Small capacities introduce variations in the effective self and mutual inductances which change as the square of the frequency.

(b) Corresponding variations are introduced in the effective resistances of the coils.

(c) The mutual inductance is no longer "pure," that is, the secondary E.M.F. is no longer in exact quadrature with the primary current. The primary and secondary circuits behave as if they have a resistance in common which varies as the square of the frequency.

(d) Eddy currents produce impurity and changes of effective resistance which are initially of the same nature as those due to capacity.

(e) Eddy current and capacity effects may be so combined that the variations of mutual inductance and of impurity cancel simultaneously, so that it is theoretically possible to obtain a "pure" mutual inductance having no variation with frequency over the whole range of audio frequencies.

2. The conditions of balance of three alternating-current bridges, viz.:—

(a) The Heaviside inductance bridge.

(b) The Carey-Foster bridge.

(c) A modified Campbell frequency bridge

have been worked out on the basis that the mutual inductance is not pure and the mutual and self-inductances are not invariable with frequency. It is shown how combined observations on these bridges may be used to determine the frequency corrections of a given inductometer.

3. An experimental study has been made upon an inductometer of the Campbell type. The results show that if such an inductometer is used at the higher audio-frequencies for the measurements of the effective resistance of coils or of condensers, the inductometer corrections may in certain cases be larger than the quantities measured.

4. A method for automatically correcting an inductometer is suggested in order that it may be used for the measurement of power factors of condensers without having recourse to tables of corrections.

1.

It is usually assumed when mutual inductances are employed in alternating-current bridge measurements that the E.M.F. developed in the secondary coil is in exact quadrature



with the primary current. This assumption is practically correct at low frequencies, but at the higher telephonic frequencies a component of secondary E.M.F. which is in phase with the primary current begins to become appreciable.

Mutual inductances in which this in-phase component is developed are said to be "impure," and the magnitude of the in-phase component when unit current flows in the primary may be taken as the measure of the impurity.\* Impurity, as defined in this way, is of the dimensions of a resistance, and if it is denoted by  $\sigma$ , the vector relation connecting secondary E.M.F.  $E_s$  with primary current  $I_p$  must be written

$$E_s = (\sigma + j\omega M)I_p, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$M$  being the mutual inductance,  $\omega/2\pi$  the frequency and  $j$  the operator rotating through a right angle.

## 2. Factors Producing Impurity.

In the most general case, the factors bringing about impurity in a mutual inductance composed of coils with air cores are self and intercapacities, leakage, and eddy-current losses. If the coils have a common point, impurity might also arise due to a small resistance in common with the primary and secondary circuit, unless proper precautions are taken in regard to the mode of connection. The Carey-Foster bridge used in measurements of the capacity and effective resistance of condensers is particularly liable to error owing to this form of impurity, as the common point of the mutual inductance is a four branch point. (See Fig. 7.) Leakage is usually of little importance with proper insulation. However, a non-uniform distribution of resistance or inductance, combined with distributed capacity, may cause an inductance to behave as if it possessed an end to end capacity having leakage. This is readily seen by considering the case of a pure inductance,  $L$ , shunted by a capacity,  $C$ , and connected in series with a resistance,  $R$ . If  $C$  and  $R/\omega L$  are small, then the capacity  $C$  across  $L$  alone is equivalent to a capacity  $C$ , having leakage  $-2CR/L$ , and acting as a shunt across both  $R$  and  $L$ .

The effects of self and intercapacity are treated in section 4 for the case where the two coils have a common point. It is there shown that small capacities produce impurities varying as the square of the frequency, and this variation is accom-

\* Silsbee, Bull. Bureau of Standards, Vol. XIII., p. 414, 1916.

panied by variations of effective resistance and inductance following the same law.

The effects due to eddy currents are considered in section 7 *et seq.* The effects are mainly variations in impurity and effective resistance, the changes in inductance due to eddy currents being small. If the frequency is not too high, the changes in impurity and effective resistance are proportional to the square of the frequency. Thus it is impossible by effective resistance measurements alone to separate the effects due to capacity from those due to eddy currents, but some estimate of the relative importance of the two effects might be obtained by determining the frequency coefficients of the inductances and assuming these to be due mainly to capacity.

### 3. Preliminary Network Transformations.

The theory to be given is much simplified if use is made of the following network transformations.\*

(A) Two inductive coils (Fig. 1a), with resistance operators

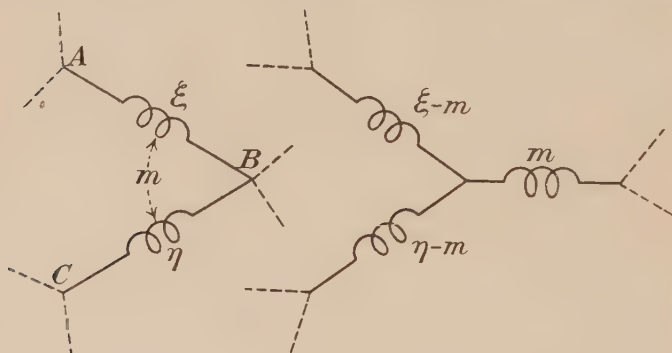


FIG. 1a.

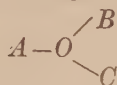
FIG. 1b.

$\xi$ ,  $\eta$  and mutual operator  $m$ , and having a common point, may be replaced by three inductive coils having resistance operators  $\xi-m$ ,  $\eta-m$ ,  $m$ , and no mutual operators radiating as a "star" combination from the common point. (Fig. 1b.)

In this transformation,  $m$  must be taken as positive when the directions of winding are such that if current is applied to the combination at the open ends the mutual inductance will oppose the self-inductances.

\* G. A. Campbell, Trans. Am.I.E.E., Vol. XXX., p. 891, 1911.

(B) Three arms,  $OA$ ,  $OB$ ,  $OC$ , having resistance operators  $a$ ,  $b$ ,  $c$  respectively, and forming a "star" combination



may be replaced by a triangular "mesh,"



provided that the resistance operators  $X$ ,  $Y$ ,  $Z$  of the arms  $BC$ ,  $CA$ ,  $AB$  have the values  $(bc+ca+ab)/a$ ,  $b$ ,  $c$ .

Conversely, the mesh may be replaced by a star, and then

$$a, b, c = (YZ, ZX, XY) / (X + Y + Z).$$

If the symbols represent *conductance* operators, the above relations are interchanged.

#### 4. The Effect of Self and Intercapacities in Mutual Inductances.

Let the two coils have a common point. Let their resistance operators, apart from capacity, be  $\xi$ ,  $\eta$ , and let their mutual operator be  $m$ . Suppose the self capacities to be sufficiently represented by condensers of capacities  $c_1$ ,  $c_2$  in parallel with the coils  $\xi$ ,  $\eta$ , and let the intercapacity be represented by a condenser of capacity  $c_{12}$ , joining the open ends of the coils. The system is then as shown in Fig. 2a.

By application of transformation (A), the system is converted to that of Fig. 2b, in which

$$x = m, y = \xi - m, z = \eta - m \quad \dots \dots \dots (2)$$

and  $a$ ,  $\beta$ ,  $\gamma$  are the conductance operators of the capacity arms  $c_{12}$ ,  $c_2$ ,  $c_1$ .

By application of transformation (B) to the star combination  $x$ ,  $y$ ,  $z$ , the system of Fig. 2b is converted to that of Fig. 2c, in which

$$\begin{aligned} a' &= a + x/\mu, \beta' = \beta + y/\mu, \gamma' = \gamma + z/\mu \\ \mu &= yz + zx + xy \quad \dots \dots \dots (3) \end{aligned}$$

By the converse of transformation (B), the system of Fig 2d is then obtained, in which

$$\left. \begin{aligned} X, Y, Z &= \frac{x + a\mu, y + \beta\mu, z + \gamma\mu}{1 + a(y + z) + \beta(z + x) + \gamma(x + y) + \lambda\mu} \dots \dots (4) \\ \lambda &= \beta\gamma + \gamma a + a\beta \end{aligned} \right\}$$



Finally, by the converse of transformation (A), the system of Fig. 2*d* is converted to that of Fig. 2*e*, in which

$$\xi' = X + Y, \eta' = Z + X, m' = X \quad . \quad . \quad . \quad (5)$$

The result is thus a system in which the capacities have been suppressed, so that  $\xi'$ ,  $\eta'$ ,  $m'$  are the modified operators to employ to include the effects of capacity. If  $c_1$ ,  $c_2$ ,  $c_{12}$  are

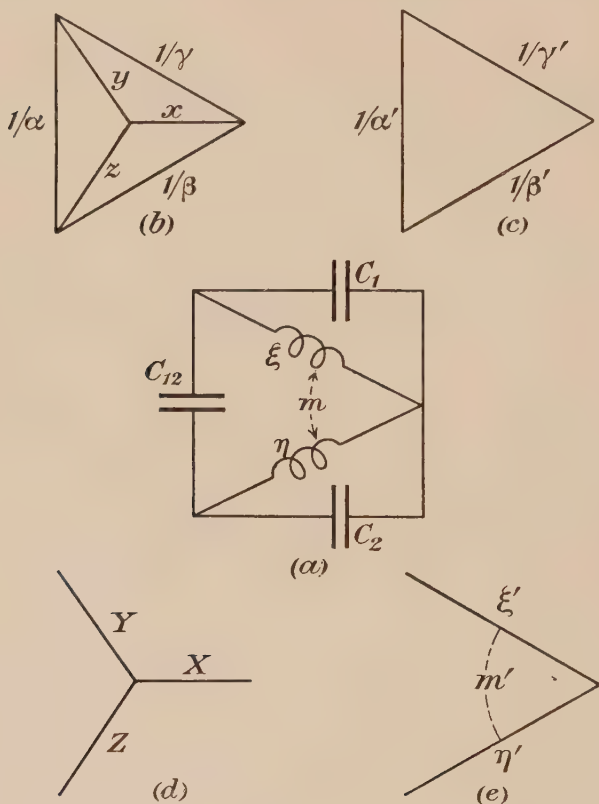


FIG. 2.

small, quantities involving their squares or products may be neglected, and in this case it is found from (2), (4) and (5) that

$$\left. \begin{aligned} m' &= m + \alpha(\xi - m)(\eta - m) - \beta m \eta - \gamma m \xi \\ \xi' &= \xi - \alpha(\xi - m)^2 - \beta m^2 - \gamma \xi^2 \\ \eta' &= \eta - \alpha(\eta - m)^2 - \beta \eta^2 - \gamma m^2 \end{aligned} \right\} . \quad . \quad . \quad (6)$$

Now, in the case of currents alternating with frequency  $\omega/2\pi$ , resistance operators may be replaced by vector impedances, so that we may then write

$$\left. \begin{aligned} \xi &= R_1 + j\omega L_1, & \eta &= R_2 + j\omega L_2, & m &= j\omega M \\ a &= j\omega C_{12}, & \beta &= j\omega C_2, & \gamma &= j\omega C_1 \end{aligned} \right\}, \quad \dots \quad (7)^*$$

in which  $R_1, R_2$  are the resistances of the coils,  $L_1, L_2, M$  the inductances.

Applying (7) to (6) and separating the real and imaginary parts, the following formulæ are obtained.

$$m_e = C_{12}R_1R_2 + \omega^2 \{C_1L_1M + C_2L_2M - C_{12}(L_1 - M)(L_2 - M)\} \quad (8)$$

$$\sigma_e = \omega^2 [C_1R_1M + C_2R_2M - C_{12}\{R_1(L_2 - M) + R_2(L_1 - M)\}]. \quad (9)$$

$$l_{1e} = -(C_1 + C_{12})R_1^2 + \omega^2 \{C_1L_1^2 + C_2M^2 + C_{12}(L_1 - M)^2\} \quad (10)$$

$$l_{2e} = -(C_2 + C_{12})R_2^2 + \omega^2 \{C_1M^2 + C_2L_2^2 + C_{12}(L_2 - M)^2\} \quad (11)$$

$$r_{1e} = 2\omega^2 R_1 \{C_1L_1 + C_{12}(L_1 - M)\} \quad \dots \quad (12)$$

$$r_{2e} = 2\omega^2 R_2 \{C_2L_2 + C_{12}(L_2 - M)\} \quad \dots \quad (13)$$

In these formulæ  $m_e, l_{1e}, l_{2e}$  are the increases in mutual and self inductance,  $\sigma_e$  is the impurity introduced by the capacities,  $r_{1e}, r_{2e}$  are the added resistances due to capacity.

### 5. Measurement of Self and Intercapacity.

If it may be assumed that the variations of self and mutual inductance are due to capacity only, equations (8), (10), and (11) may be used to determine the capacities  $C_1, C_2, C_{12}$ .

With both coils in position and having a common point, the effective self inductances of the two coils separately and in series are determined at a number of different frequencies. Then, if for brevity we denote the coefficients of  $\omega^2$  in (8), (10), (11) by  $k_{12}, k_1, k_2$  and express  $C_1, C_2, C_{12}$  in terms of these coefficients, we obtain

$$\left. \begin{aligned} (L_1L_2 - M^2)^2 C_1 &= k_1 L_2 (L_1 - M) - k_2 M (L_1 - M) \\ &\quad + k_{12} (L_1L_2 - 2ML_1 + M^2) \\ (L_1L_2 - M^2)^2 C_2 &= -k_1 M (L_2 - M) + k_2 L_1 (L_2 - M) \\ &\quad + k_{12} (L_1L_2 - 2ML_2 + M^2) \\ (L_1L_2 - M^2)^2 C_{12} &= k_1 ML_2 + k_2 ML_1 - k_{12} (L_1L_2 - M^2) \end{aligned} \right\} \quad (14)$$

\* If leakage is also present  $\alpha, \beta, \gamma$  are of the forms  $G + j\omega C$  in which  $G$  is the leakance associated with the capacity  $C$ .

These equations determine  $C_1$ ,  $C_2$ ,  $C_{12}$  as  $k_1$ ,  $k_2$ ,  $k_{12}$  and may be deduced from the frequency variations of the self and mutual inductances.

The method fails when the coupling is very close, as, for perfect coupling, the three relations employed are no longer independent.

The values of  $C_1$ ,  $C_2$  obtained by this method are not necessarily the same as would be obtained when the coils are separated from each other, as the capacities  $C_1$ ,  $C_2$  include any intercapacity that may be regarded as acting from the inner end of one coil to the outer end of the other. Also if the common point is altered the values of  $C_1$ ,  $C_2$ ,  $C_{12}$  may

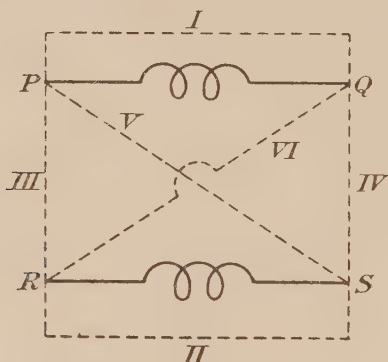


FIG. 3.

also change because of the redistribution of the intercapacities.

However, by a systematic alteration of the common points the assumption that distributed intercapacities may be replaced by end capacities may be put to an experimental test.

Thus, when the coils are not connected the system is as in Fig. 3, in which the broken lines represent the capacity arms. Four common points are possible, so that 12 combinations of the six capacities—I, II, III, IV, V, VI—may be measured. These are given in the following table:—

Common Point.	$C_1$		$C_2$		$C_{12}$
PR .....	I+VI	...	II+V	...	IV
QS .....	I+V	...	II+VI	...	III
QR .....	I+III	...	II+IV	...	V
PS .....	I+IV	...	II+III	...	VI



Since the values of  $C_{12}$  give the four intercapacities, the remaining observations give four independent determinations of the self capacities I and II respectively. If the end to end capacity theory is adequate, these should be consistent and in agreement with the self capacities as determined when the coils are separated.

6. *Conditions for Zero Impurity and for No Variation of Mutual Inductance with Frequency.*

By equation (8) the frequency coefficient of mutual inductance is zero when

$$C_{12} = M(C_1 L_1 + C_2 L_2) / (L_1 - M)(L_2 - M). \quad (15)$$

By equation (9) the impurity due to capacity is zero when

$$C_{12} = M(C_1 R_1 + C_2 R_2) / \{R_1(L_2 - M) + R_2(L_1 - M)\}. \quad (16)$$

If condition (15) is inserted in equation (9) an essentially negative value is obtained for  $\sigma_c$ , so that a mutual inductance invariable with frequency is always associated with a negative capacity impurity. As, however, the whole impurity includes both capacity and eddy current terms, and as the latter may be positive or negative it is theoretically possible to construct a mutual inductance invariable with frequency and free from impurity. This conclusion only holds when the common point is such that the two coils in series have their minimum inductance.

The procedure indicated by theory to bring about this result is to secure the condition of invariability of mutual inductance by artificial addition of self or intercapacity and then the condition of zero impurity by introduction of artificial eddy losses.

EFFECTIVE RESISTANCES AND IMPURITIES DUE TO EDDY CURRENT LOSSES.

7. *Eddy Current Losses in a Long Cylinder.*

Let a long non-magnetic cylinder of diameter  $d$  and of material of resistivity  $\rho$  be placed in a uniform field alternating with frequency  $\omega/2\pi$ . Let  $H'$  and  $H$  be the maximum values of the components of the field along and perpendicular to the axis of the cylinder. Also write

$$z^2 = \pi \omega d^2 / \rho. \quad (17)$$

Then it may be shown that the rate of dissipation of energy in unit length of the cylinder due to the eddy currents set up by the alternating field is

$$W_0 = \frac{\rho}{2\pi} G(z) (H^2 + \frac{1}{2} H'^2) \quad . \quad . \quad . \quad . \quad (18)$$

in which for brevity  $G(z)$  is written for

$$\frac{1}{4} z \frac{\text{ber } z \text{ ber}' z + \text{bei } z \text{ bei}' z}{\text{ber}^2 z + \text{bei}^2 z} \quad . \quad . \quad . \quad . \quad (19)$$

When  $z$  is less than 1

$$G(z) \doteq z^4/64, \quad . \quad . \quad . \quad . \quad (20)$$

When  $z$  is greater than 4

$$G(z) \doteq (\sqrt{2}z - 1)^4/8, \quad . \quad . \quad . \quad . \quad (21)$$

while generally the following table may be used:—

$z=0.5$	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$G(z)=0.000975$	0.01519	0.0691	0.1724	0.295	0.405	0.499	0.584
$z=4.5$	5.0						
$G(z)=0.669$	0.755						

### 8. *Eddy Current Losses in a Coil.*

Equation (18) may be applied to determine the eddy current losses in a coil placed in an alternating magnetic field, provided that the values of  $H$  and  $H'$  are practically constant over the section of an individual wire, that their variation is slow along the axis of the wire and that the radius of curvature of the wire is large compared with the radius of the wire itself. Then assuming for simplicity that the field is acting transversely so that  $H'=0$ , the rate of energy dissipation in the whole coil is

$$W = \frac{\rho}{2\pi} G(z) \int_0^l H^2 dl, \quad . \quad . \quad . \quad . \quad (22)$$

where the integration is taken throughout the length  $l$  of the wire of the coil.

If  $H_m^2$  is the mean square value of the transverse field acting on the coil and  $R$  is the direct current resistance of the coil, equation (22) may be written

$$W = \frac{1}{8} d^2 R G(z) H_m^2 \quad . \quad . \quad . \quad . \quad (23)$$

### 9. *Coil in Presence of Two Other Coils.*

Let the field  $H$  be due to two currents of vector values,  $\mathbf{I}_1$ ,  $\mathbf{I}_2$ , and of phase difference  $\psi$ , flowing in two neighbouring coils. The vector fields due to  $\mathbf{I}_1$  and  $\mathbf{I}_2$ , which act on an element of the coil, will be of the forms  $a_1\mathbf{I}_1$ ,  $a_2\mathbf{I}_2$ , and will be inclined in space at an angle  $\varphi$ .

The components of the field along and perpendicular to  $a_1\mathbf{I}_1$  are

$$\mathbf{H}_1 = a_1\mathbf{I}_1 + a_2\mathbf{I}_2 \cos \varphi$$

$$\mathbf{H}_2 = a_2\mathbf{I}_2 \sin \varphi,$$

so that if  $H_1$ ,  $H_2$  are the amplitudes of  $\mathbf{H}_1$ ,  $\mathbf{H}_2$ ,  $I_1$ ,  $I_2$  the amplitudes of  $\mathbf{I}_1$ ,  $\mathbf{I}_2$ , the value of  $H^2$  is

$$\begin{aligned} H^2 &= H_1^2 + H_2^2 \\ &= a_1^2 I_1^2 + a_2^2 I_2^2 \cos^2 \varphi + 2a_1 a_2 I_1 I_2 \cos \varphi \cos \psi + a_2^2 I_2^2 \sin^2 \varphi \end{aligned}$$

and its mean value throughout the whole coil is

$$H_m^2 = \frac{I_1^2}{l} \int_0^l a_1^2 dl + \frac{I_2^2}{l} \int_0^l a_2^2 dl + 2 \frac{I_1 I_2}{l} \cos \psi \int_0^l a_1 a_2 \cos \varphi dl. \quad (24)$$

The rate of dissipation of energy in the coil is therefore obtained by using this value of  $H_m^2$  in (23).

### 10. *Resistance System Imitating Eddy Current Losses.*

Let the two coils ( $I$  and  $II$ ) which carry the currents  $I_1$  and  $I_2$  have their circuits connected by the resistance system shown in Fig. 4. The rate of dissipation of energy in this resistance system is

$$W' = \frac{1}{2} r_{1e} I_1^2 + \frac{1}{2} r_{2e} I_2^2 - \sigma_e I_1 I_2 \cos \psi, \quad \dots \quad (25)$$

and the system will dissipate the same energy as the eddy currents in the coil just considered provided that

$$\left. \begin{aligned} r_{1e} &= \frac{1}{4} d^2 R G(z) \left\{ \frac{1}{l} \int_0^l a_1^2 dl \right\} \\ r_{2e} &= \frac{1}{4} d^2 R G(z) \left\{ \frac{1}{l} \int_0^l a_2^2 dl \right\} \\ \sigma_e &= -\frac{1}{4} d^2 R G(z) \left\{ \frac{1}{l} \int_0^l a_1 a_2 \cos \varphi dl \right\} \end{aligned} \right\} \quad \dots \quad (26)$$

If by the converse of transformation (A) we convert  $\sigma_e$  to a mutual operator, the common point may be removed and  $r_{1e}$ ,



$r_{2e}$ ,  $r_e$  will then represent the increments of resistances and the impurity introduced by the eddy current losses in a neighbouring coil. If there are a number of neighbouring coils, then since the losses are additive, the eddy current resistances and impurities are sums of expressions such as (26). In particular the neighbouring coils may simply be the coils *I* and *II* and the eddy current resistances and impurities are got by performing the integrations in (26) throughout the lengths of wire in the two coils.

The increments in resistance obtained in this way are to be added to the ordinary "skin" increments of resistance, the

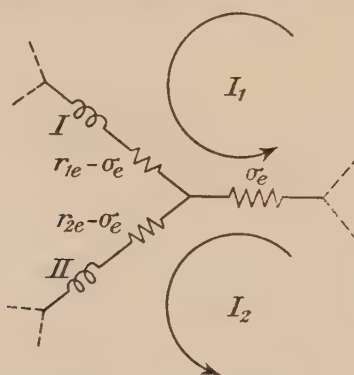


FIG. 4.

latter being due to the currents flowing in the individual elements of the coil.

#### 11. Conditions Governing Sign of $\sigma_e$ .

In applying equation (26) some care is required in order to avoid the wrong sign for  $\sigma_e$ . In Fig. 4, from which the equations have been deduced, the directions of the currents are such that if  $I_1 = I_2 = I$  the current  $I$  will flow only through the resistances  $r_{1e} - \sigma_e$ ,  $r_{2e} - \sigma_e$ , so that when the coils are connected in series the total eddy current resistance is  $r_{1e} + r_{2e} - 2\sigma_e$ . In order therefore to make  $\sigma_e$  consistent with the defining equation (1) the mutual inductance  $M$  between the coils *I* and *II* must, under the same conditions, oppose the self-inductances. Now  $a_1$ ,  $a_2$  are fields due to unit currents in coils *I* and *II* respectively, so that in order to obtain the correct sign for  $\sigma_e$  the directions of these unit

currents must be so chosen that the mutual inductance opposes the self-inductances. With this convention  $\sigma_e$  will be positive if the sum of the eddy current resistances is less than the eddy current resistance of the coils in series and helping.

## 12.

It is readily seen from equations (26) that the inequalities

$$r_{1e} + r_{2e} > 2\sigma_e$$

$$r_{1e}r_{2e} > \sigma_e^2$$

hold for the eddy current resistances, relations analogous to those between self and mutual inductances.

13. *Systems of Parallel Wires.*

The theory finds its simplest application when the coils *I* and *II* consist of systems of parallel wires. The results in these cases may be applied to coaxial coils provided that winding

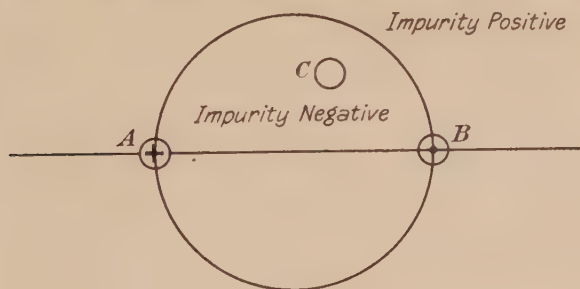


FIG. 5.

sections of the coils and their distance apart are small compared with their radii.

In order to illustrate the properties of the impurity  $\sigma_e$ , consider three parallel wires *A*, *B*, *C* (Fig. 5). Let *A* be the coil *I* and *B* the coil *II*. Let *C* carry no current. The impurity introduced by the presence of *C* will be negative, zero or positive according as the angle of inclination of the fields at *C* produced by opposite currents in *A* and *B* is less than, equal to, or greater than  $\pi/2$ . If a circle be drawn with *AB* as diameter, then since the two fields are normal at all points on this circle, the impurity due to the presence of *C* changes sign from negative to positive as *C* is moved out

of the circle. Along the line  $AB$ , the fields either help or oppose each other, so that when  $C$  is collinear with  $AB$  the condition  $r_{1e}r_{2e}-\sigma_e^2=0$  is satisfied. This is a case of perfect eddy current coupling.

The above conclusions still apply when the wires  $A$  and  $B$  are replaced by bundles of wires, provided that the distance of  $C$  is sufficient to warrant the assumption that the currents in  $A$  and  $B$  act as if concentrated at a point. It is thus possible to counteract an already existing impurity of any sign by introducing a third coil carrying no current in the proper position with respect to the two coils forming the mutual inductance. An example of this use of eddy currents to correct impurity is given in Section 19.

#### 14. System of Four Wires.

Let there be four coplanar wires, two constituting the coil  $I$  and two constituting the coil  $II$ , spaced at equal distances  $D$ . Numbering the wires in order 1, 2, 3, 4, there are three possible combinations, (1, 2), (3, 4); (1, 3), (2, 4); (1, 4), (2, 3), each giving different effective resistances and impurities.

By a simple application of the formula  $H=\frac{2I}{r}$  we obtain the following values for  $a_1$ ,  $a_2$ ,  $\cos \varphi$  for the combination (1, 2), (3, 4).

Wire	$a_1 D$	$a_2 D$	$\cos \varphi$
1	2	$5/3$	-1
2	2	3	+1
3	3	2	+1
4	$5/3$	2	-1

Using these values in (26) we find

$$r_{1e}=2.472 \frac{d^2}{D^2} G(z)R=r_{2e}, \sigma_e=-0.667 \frac{d^2}{D^2} G(z)R$$

in which  $R$  is the direct-current resistance of one pair of wires. By similar methods for the combination (1, 3), (2, 4) we find

$$r_{1e}=1.139 \frac{d^2}{D^2} G(z)R=r_{2e}, \sigma_e=+0.667 \frac{d^2}{D^2} G(z)R$$

and for the combination (1, 4), (2, 3)

$$r_{1e}=0.361 \frac{d^2}{D^2} G(z)R, r_{2e}=3.250 \frac{d^2}{D^2} G(z)R, \sigma_e=0.$$

In all three cases  $r_{1e} + r_{2e} + 2\sigma_e$  has the same value, viz.,  $3.611 \frac{d^2}{D^2} G(z)R$ , which is, of course, the proper value for the proximity resistance of a system of four parallel wires carrying currents in the same direction. The example also shows that the sign and magnitude of the impurity depends upon the interlacing of the wires of the two coils, the tendency being for the impurity to increase positively as the interlacing increases. This point may be further illustrated by a system of 12 equidistant wires, of which six constitute one circuit and six the other circuit. If we write

$$\sigma_e = u_{12} \frac{d^2}{D^2} G(z)R,$$

$u_{12}$  takes the following values for different arrangements:—

Arrangement	$u_{12}$
1 1 1 1 1 2 2 2 2 2 2 2	—0.80
1 1 1 1 1 2 2 2 2 2 2 1	—0.70
1 1 1 1 2 2 2 2 2 2 2 1	—0.66
1 1 1 2 2 2 2 2 2 1 1 1	—0.65
1 1 1 2 2 2 1 1 1 2 2 2	—0.35
1 1 2 2 1 1 2 2 1 1 2 2	—0.00 <sub>3</sub>
1 2 1 2 1 2 1 2 1 2 1 2	+0.66

Interlacing the coils will increase intercapacity effects, so that if the coils are used with their common point such that the inductance in series is a minimum, the effect of increasing intercapacity will cause a capacity impurity acting oppositely to the eddy current impurity.

#### 15. *Short Single Layer Coils in Contact.*

The eddy current impurity between two short coils of equal radius in end contact may be calculated if their separate effective resistances  $r_{1e}$ ,  $r_{2e}$  and their effective resistance  $r_s$  in series are known for  $r_s = r_{1e} + r_{2e} + 2\sigma_e$ .

The value of  $r_s$  has been shown by the author in another Paper to be of the form

$$r_s = u_n \frac{d^2}{D^2} G(z)R. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Where  $R$  is the direct-current resistance of the whole combination,  $G(z)$  is defined in Section 7,  $d$  is the diameter of the wire,  $D$  the distance of consecutive wires and  $u_n$  is a



factor depending on the number of wires  $n$  in the two coils. The following values when plotted against  $1/n$  enable  $u_n$  to be determined in any case:—

$$\begin{array}{cccccc} n = & 1 & 2 & 3 & 5 & 10 & \text{inf.} \\ u_n = & 0.000, & 1.000, & 1.500, & 2.014, & 2.514, & 3.290 \end{array}$$

The values of  $r_{1e}$ ,  $r_{2e}$  are, however, greater than would be estimated by the above method because of the eddy currents induced in one coil by the current in the other. If the number of wires is finite in both coils, the additive term due to the presence of the other coil may be calculated by methods similar in principle to those employed in the case of the four-wire system. When the number of wires is large, this method becomes laborious, but the extreme case of an infinite number of wires may be obtained by integration. Treating the narrow coils as straight-wire systems, the field at any point in the plane of a strip of width  $a$  is  $2 \frac{I}{D} \log \frac{a+x}{x}$  where  $I$  is the current in each wire and  $x$  the distance of the point from the edge of the strip. If the strip representing the other coil has width  $b$ , the mean square value of the field over this coil is

$$\frac{4I^2}{D^2} \cdot \frac{1}{b} \int_0^b \log^2 \frac{a+x}{x} dx = \frac{4I^2}{D^2} \frac{a}{b} f\left(\frac{b}{a}\right)$$

in which 
$$f\left(\frac{b}{a}\right) = \int_0^{\frac{b}{a}} \log^2 \frac{1+\mu}{\mu} d\mu.$$

The values of  $f(\mu)$  may be obtained in the form of series. Thus suitable series are:

$$\mu < 2$$

$$f(\mu) = -\mu \log^2 \nu + 2 \log \nu \log(1-\nu) + 2 \sum_{r=1}^{\infty} \frac{\nu^r}{r^2} \quad \dots \quad (28)$$

in which

$$\nu = \mu/(1+\mu);$$

$$\mu > 2$$

$$f(\mu) = \frac{\pi^2}{3} - \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r \mu^r} \sum_{s=1}^{\infty} \frac{1}{s(s-r+1)} \quad \dots \quad (29)$$

By (26) the term in  $r_{1e}$  for the coil (a), which is due to the eddy currents in (b) is

$$\frac{d^2}{D^2} G(z) \frac{a}{b} f\left(\frac{b}{a}\right) R = \frac{d^2}{D^2} G(z) f\left(\frac{b}{a}\right) R_a,$$

where  $R_a$ ,  $R_b$  are the direct-current resistances of  $R_a$  and  $R_b$ .

Hence

$$\left. \begin{aligned} r_{1e} &= \left\{ u_n(a) + f\left(\frac{b}{a}\right) \right\} \frac{d^2}{D^2} G(z) R_a \\ r_{2e} &= \left\{ u_n(b) + f\left(\frac{a}{b}\right) \right\} \frac{d^2}{D^2} G(z) R_b \end{aligned} \right\} \dots (30)$$

These equations, together with (27) and the tables of  $u_n$  and  $f(b/a)$  enable the eddy current resistances and impurities of short single layer coils in end contact to be calculated. In the tables for  $f(b/a)$  the values for the case  $b/a$  infinite but  $n$  finite are calculated from the formula

$$f\left(\frac{b}{a}\right) = \frac{\pi^2}{6} + 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(n-1)^2} - \frac{1}{n} \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} \right).$$

Where  $n$  is the number of wires in  $a$ .

When  $n$  is also infinite  $f(b/a) = \pi^2/3$ .

*Values of  $f(b/a)$  in Equations (30).*

$n_a$  = number of wires in system  $a$ .

$n_b$  = " " " " "  $b$ .

$b/a < 1$ .

$n_a$	$n_b$	$f(b/a)$	$n_a$	$n_b$	$f(b/a)$ .
2	1	1.125	5	1	1.042
3	2	1.511	10	4	1.704
3	1	1.120	10	3	1.523
5	4	1.858	10	2	1.265
5	3	1.701	10	1	0.858
5	2	1.462			

$n_a = n_b = \text{infinity}$ .

$b/a = 0.2 \quad 0.4 \quad 0.6 \quad 0.8$

$f(b/a) = 1.375 \quad 2.089 \quad 2.334 \quad 2.492$

$b/a > 1$ .

$n_a$	$b/a = 1$	2	3	4	5	inf.
1	1.000	1.250	1.361	1.423	1.463	1.645
2	1.472	1.743	1.859	1.923	1.963	2.145
3	1.715	1.992	2.108	2.172	2.209	2.395
5	1.970	2.248	2.365	2.429	2.470	2.652
10	2.218	2.497	2.613	2.674	2.710	2.901
inf.	2.606	2.886	3.002	3.067	3.119	3.290

For interpolation  $f(b/a)$  may be plotted against  $b/a$  for different values of  $n_a$ , and against  $1/n$  for different values of  $b/a$ .

16.

As an illustration of the application of these results let the coil (*a*) have 20 turns, and the coil (*b*) 30 turns.

Then by graphical interpolation from the tables we find for coil (*a*)

$$f(b/a, n_a) = f(3/2, 30) = 2.55$$

$$u_n(a) = u_{20} = 2.84,$$

so that by (30)

$$r_{1e} = 5.39 \times 20 G(z) \frac{d^2}{D^2} R, \quad \dots \quad (A)$$

*R* being the direct-current resistance of one turn of wire.

For coil (*b*)

$$f(a/b, n_b) = f(2/3, 30) = 2.24$$

$$u_n(b) = u_{30} = 2.98$$

and

$$r_{2e} = 5.22 \times 30 G(z) \frac{d^2}{D^2} R. \quad \dots \quad (B)$$

For coils (*a*) and (*b*) in series, helping

$$f = 0 \quad u_n = u_{50} = 3.10$$

and

$$r_{1e} + r_{2e} + 2\sigma_e = r_s = 3.10 \times 50 G(z) \frac{d^2}{D^2} R. \quad \dots \quad (C)$$

From (A), (B) and (C)

$$r_{1e} : r_{2e} : \sigma_e = 10.8 : 15.7 : 5.5$$

$\sigma_e$  being negative.

The absolute magnitudes will depend upon the frequency, the diameter of the wire and the spacing. If we assume the wire to be of copper of resistivity 1,600 C.G.S. units, and of diameter 1 mm., then by (17) the value of *z* is unity when the frequency is approximately 8,000 cycles per second, and then  $G(z) = 0.01519$ . Therefore if we take  $d/D = 0.8$

$$\frac{r_{1e}}{20R} = 5.39 \times 0.01519 \times (0.8)^2 = 0.0523,$$

so that at this frequency the value of  $r_{1e}$  is 5.2 per cent. of the direct-current resistance of coil (*a*).

The ordinary skin effect for copper wire of diameter 1 mm.

at this frequency causes an increase of 0.52 per cent., and this must be added to the above value. If coil (*b*) had been absent the value of  $r_{1e}$  would be reduced in the ratio 2.84/5.39, so that slightly less than half of the above increase is due to the presence of the coil (*b*).

For frequencies less than 8,000 cycles per second the eddy current resistances would fall as the square of the frequency, as by section (7)  $z^2$  is proportional to frequency, and  $G(z)$  is proportional to  $z^4$ .

At greater frequencies the increase is not so rapid, and may be deduced from the values of  $G(z)$  given in section (7),  $z$  being proportional to the square root of the frequency.

In certain applications the values of  $r_{1e}/L_1\omega$ ,  $r_{2e}/L_2\omega$ ,  $\sigma_e/M\omega$  are of importance, the latter quantity giving the want of quadrature of the mutual inductance. In the above example if we suppose the coils such that  $R/L$  for coil (*a*) is about 1,000, we have  $r_{1e}/L_1\omega=0.001$ , and this also gives the order of  $\sigma_e/M\omega$ . Hence at a frequency of 8,000 cycles per second the secondary E.M.F. will lead the primary current by an angle which exceeds a right angle by about 0.001 radian, *i.e.*, about 3'.

This angle does not increase indefinitely with frequency, as from (27) and (17)  $\sigma_e/M\omega$  is proportional to  $G(z)/z^2$ . From the table of values for  $G(z)$  it is seen that  $G(z)/z^2$  is maximum when  $z=2.6$ , and (for the above example) the maximum value for the want of quadrature is about 10'.

It should be noted that for a given spacing factor this maximum angle is independent of the diameter of the wire, but is pushed further up the scale of frequency as the diameter of the wire diminishes.

### 17. *Short Many-layered Coils Wound Over Each Other.*

The investigation will be confined to the case in which the winding depth is small compared with the winding breadth, and the latter small compared with the coil radius. The results apply whether the coils are solenoidal or flat.

For a single coil of winding breadth,  $b$ , having  $m$  layers and  $n$  turns per layer of wire of diameter  $d$ , and direct-current resistance  $R_1$ , the term to be added to the effective resistance due to the increased eddy currents produced by coiling, has been shown by the author to be

$$r_e = \frac{1}{3}\pi^2 \left(\frac{nd}{b}\right)^2 (2m^2 - 1)G(z)R_1 \quad . \quad . \quad . \quad (31)$$



If a second coil of  $M$  layers is wound over this coil there is a further increase in resistance due to the eddy currents induced in the wire of the second coil.

The components of the field due to the current  $I$  in the first coil acting along and perpendicular to the winding breadth have the values  $2\pi mnI/b$ , and  $2mnI \log \frac{b-x}{x}/b$  respectively at a point on its surface distant  $x$  from the edge. If the second coil has very small winding depth, and is close on the first, the mean square value of the field having the above components may be used in (23) to determine the required additive term.

Since

$$\frac{1}{b} \int_0^b \log^2 \frac{b-x}{x} dx = \frac{\pi^2}{3},$$

the result is

$$r'_e = \frac{4}{3} \pi^2 \left( \frac{nd}{b} \right)^2 MmG(z)R_1, \quad \dots \quad (32)$$

remembering that  $R$  in (23) refers to the second coil, and is  $R_1M/m$ .

Combining with (31) we have

$$r_{1e} = \frac{1}{3} \pi^2 \left( \frac{nd}{b} \right)^2 G(z)R_1(2m^2 + 4Mm - 1). \quad \dots \quad (33)$$

Similarly

$$r_{2e} = \frac{1}{3} \pi^2 \left( \frac{nd}{b} \right)^2 G(z)R_2(2M^2 + 4Mm - 1), \quad \dots \quad (34)$$

and for the two coils in series helping (31) immediately gives

$$r_{1e} + r_{2e} + 2\sigma_e = \frac{1}{3} \pi^2 \left( \frac{nd}{b} \right)^2 G(z)(R_1 + R_2) \{2(M+m)^2 - 1\} \quad (35)$$

Solving for  $\sigma_e$  with  $\frac{R_1}{m} = \frac{R_2}{M}$ ,

$$\sigma_e = \frac{1}{3} \pi^2 \left( \frac{nd}{b} \right)^2 G(z)(R_1 + R_2)Mm. \quad \dots \quad (36)$$

Comparing with (31)  $\sigma_e/r_e = M(M+m)/(2m^2 - 1)$  from which it is seen that in the case of equal coils the impurity is slightly greater than the term due to the coiling when the two coils are separate. Since the electromagnetic coupling is fairly close, the value of  $r_e/L\omega$  for either coil may be taken as a measure also of  $\sigma_e/M\omega$ , the want of quadrature.

It should be noted also that  $\sigma_e$  is positive for coils with their widest edges in contact, thus differing from the coils in end contact treated in section 15.

## 18.

Some notion of the magnitude of the eddy current impurities in coils of the type just considered may be obtained by taking wire of a diameter of 1 mm. and a frequency of 8,000 cycles per second. Suppose also that for a single layer  $R_1/L_1=1,000$ . By the example of section (16) the value of  $r_e/L_1\omega$  for a single layer is of the order 5/10,000. Now as the layers increase  $L$  increases as  $m^2$ ,  $R_1$  as  $m$ , so that by (31)  $r_e/L_1\omega$  will increase as  $(2m^2-1)/m$ . If  $m$  is large the value of  $r_e/L_1\omega$  will therefore be  $m/1,000$ , and this may also be taken as a measure of  $\sigma_e/M\omega$ . Errors in quadrature due to this cause will therefore amount to one per cent. (34') at a frequency of 8,000 cycles per second. For lower frequencies and diameters the error is proportional to the frequency and the fourth power of the diameter, so that the formula

$$\frac{\sigma_e}{M\omega} \doteq \frac{mf}{8 \times 10^9} \cdot \frac{R}{L} d^4$$

may be used to give a very rough estimate of the want of quadrature. In this formula  $R/L$  is the time constant of a single layer,  $d$  the diameter of the wire in mm.,  $f$  the frequency in cycles per second,  $m$  the number of layers in one of the two equal coils.

Separation of the coils will not necessarily determine the value of  $\sigma_e/M$  as both  $\sigma_e$  and  $M$  will diminish.

### 19. *Balancing of Impurity and Frequency Co-efficient of a Mutual Inductance.*

It will now be shown that it is possible to introduce inter-capacity and eddy current losses of such a magnitude as to balance simultaneously an already existing frequency co-efficient and impurity in a mutual inductance, provided that the inductance is used with a common point in such a way that the coils are in opposition when in series. Let the two coils be equal and coaxial. Let the radius of each coil be 10 cm., their separation 3 cm. and let there be 240 turns with a square winding section of 1 cm. side. Each coil would then have an inductance of approximately 24 millihenries

and their mutual inductance would be approximately 10 millihenries. The frequency co-efficient of mutual inductance will be assumed to be positive and of amount  $4/1,000$  at a frequency of 2,000 cycles per second.\* The impurity at the same frequency will be taken to be  $+0.08$  ohm.\*

If intercapacity is introduced then by (8) and (9) with  $L_1 = L_2 = L$ , the frequency co-efficient of  $M$  is reduced by  $\omega^2 C_{12} (L - M)^2 / M$  and the impurity by  $2R\omega^2 C_{12} (L - M)$ . Using the above values of  $L$ ,  $M$  the value of  $C_{12}$  required to wipe out the already existing frequency co-efficient is approximately  $1,280 \mu\mu F$ . This will also introduce negative impurity of amount  $0.0057R$  ohms. If the resistance of the coils is less than 14 ohms resistance may be added until  $0.0057R = 0.08$ , and this will be sufficient to balance the impurity.

If  $R$  is greater than 14 ohms, the impurity has become negative by the addition of capacity, and in this case an auxiliary coil of the same radius as the other coils can be placed coaxially with them to introduce positive impurity. Thus suppose the auxiliary coil to consist of wire 1 mm. diameter placed at 1.5 cm. separation from the nearer coil, the positive impurity

introduced by the auxiliary coil is  $\frac{N^2 d^2}{D_1 D_2} G(z) R'$ ,

where  $N$  = No. of turns in the inductometer coils = 240.

$a$  = diameter of wire = 0.1 cm.

$D_1, D_2$  = separations of coil from inductometer coils.  
= 1.5, 4.5 cm.

$R'$  = resistance of auxiliary coil.

Using these values and remembering that  $G(z) = 0.01519$  at a frequency of 8,000 cycles per second and falls as the square of the frequency, the value of  $R'$  to give 0.01 ohm impurity at 2,000 cycles per second is

$$R' = 0.124 \text{ ohm.}$$

This corresponds to about 10 turns of wire of radius 10 cm.

This balancing will hold over the range of frequencies, for which the square law holds. If the diameter of any wire used does not exceed 1 mm. the range is up to 8,000 cycles per second.

\* These are the order of magnitude of these quantities in the Campbell type of inductometer.

## 20. *Effect of Impurity, &c., in Alternating Current Bridges.*

The effect of impurity in alternating current bridge measurements in which mutual inductances are employed may be included by replacing the mutual impedance  $j\omega M$  in the vector equations of balance of these bridges by  $\alpha + j\omega M$ . In interpreting these equations account should of course be taken of the variations of inductance and resistance of the various arms with frequency. Three bridges will be considered.

- (1) The Heaviside Inductance bridge with equal arms.
- (2) The Carey Foster Inductance-capacity bridge.
- (3) A modified Campbell frequency bridge.

## 21. *The Heaviside Inductance Bridge.\**

The bridge is shown in Fig. 6.  $P, P$  are two similar coils of small residual inductance.  $L, R$  is the inductive coil to be

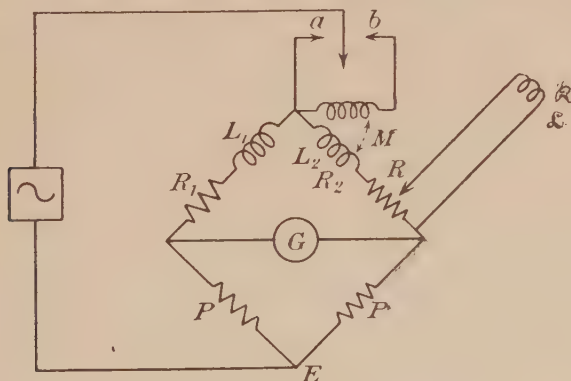


FIG. 6.—HEAVISIDE INDUCTANCE BRIDGE.

measured by means of the inductometer  $M$ .  $R$  is a constant inductance rheostat.

Let the procedure be as follows :—

With the source connected to  $a$  and  $\mathcal{L}, \mathcal{R}$  removed,  $L_1$  and  $R$  are adjusted for balance. Then with  $\mathcal{L}, \mathcal{R}$  in position and the source connected to  $b$ ,  $M$  and  $R$  are adjusted for balance. Let the two values of  $R$  be  $R', R''$ .

If residuals and impurities are ignored  $\mathcal{L} = 2M$  and  $\mathcal{R} = R' - R''$ .

As regards residuals, if observations are repeated with the

\* Heaviside, "Phil. Mag.," p. 173, Vol. XXIII., 1887.



arms  $P$ ,  $P$  reversed the residuals of  $P$ ,  $P$  cancel in the mean. Also,  $R$  being of constant inductance the residuals of the other arms cancel when the above procedure is used.

If the mutual inductance has impurity  $\sigma$  a simple application of transformation (A) shows that the two conditions of balance are  $\mathcal{Q}=2M$ ,  $\mathcal{R}=2\sigma+R'-R''$

It should be noted that the value of  $R_2$  for the coil  $L_2$  only remains the same in the two adjustments provided that the inductive action of  $M$  is introduced by the method of transference of the source from  $a$  to  $b$  *keeping the primary coil of the inductometer in position*. This is the procedure used in the Campbell form of inductometer, the fine adjustment only being made by a moving coil. In inductometers in which the whole of the adjustment is made by a moving coil, the eddy current and capacity conditions may be different in the two balances, so that  $R_2$  will have different effective values.

Since for an inductometer of 10 millihenries  $\sigma$  may be of the order 0.1 ohm at a frequency of 2,000 cycles per second, neglect of  $\sigma$  in the relation between the resistance may introduce errors of order two per cent. in measurements of the effective resistance of 20 millhenry inductances of a nominal resistance of 10 ohms.

## 22. *The Carey-Foster Inductance Capacity Bridge.\**

The elements of this bridge (Fig. 7) are :—

(a) The condenser arm having effective capacity  $C$  and effective series resistance  $S$ .†

Let its vector impedance be  $Z=S-j/\omega C$ .

(b) The inductive arm  $P$ ,  $L$  having vector impedance  $\xi=P+j\omega L$ ;

(c) The resistance arm  $R$  having residual inductance  $l$  and therefore vector impedance  $Y=R+j\omega l$ ;

(d) The mutual inductance  $M$  having impurity  $\sigma$  and acting on the inductive arm. Its mutual impedance is  $m=\sigma+j\omega M$ .

The condition of balance is readily found by applying transformation (A) to the mutual inductance. The result is a Wheatstone bridge of which the upper arms have vector

\* A. Campbell, Proc. Phys. Soc., Vol. XX., p. 626 (1907). Also Phil. Mag., October, 1907.

† The effective capacity and resistance refer to the whole arm. Thus  $C$  includes the equivalent capacities of the leads and any residual inductance in  $S$ , while  $S$  includes the effective resistance of the condenser and leads as well as any added resistance.

impedances  $m$ ,  $\xi - m$ , and the lower arms have vector impedances  $Y, Z$ . The vector equation of balance is therefore

$$mZ = Y(\xi - m) \quad (37)$$

Inserting the values of  $m, \xi, Y, Z$  and separating real and imaginary parts,

$$M/C + \sigma S = (P - \sigma)R - \omega^2 l(L - M) \quad (38)$$

$$MS - \sigma \omega^2 K = R(L - M) + Pl \quad (39)$$

In these equations  $M, K, \sigma, P, L, S$  may vary with frequency. Also  $\sigma, l$  and the frequency variations may be taken to be so small that their squares or products may be ignored. Thus in

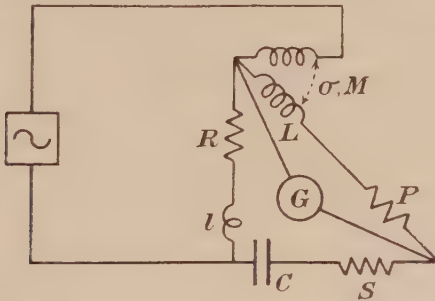


FIG. 7.—THE CAREY-FOSTER INDUCTANCE CAPACITY BRIDGE.

the small terms of (38) and (39) we may assume with sufficient accuracy that the relations

$$MC = PR, S = R(L - M)/M \quad (40)$$

will hold.

In regard to equation (38) if we write  $P = P_0 + r$ , where  $P_0$  is the direct current resistance of  $P$  and  $C_0 = M/P_0R$  we obtain to the first order of small quantities

$$C = C_0 \left\{ 1 - \frac{r - L/M}{P_0} + \frac{\omega^2 l(L - M)}{P_0 R} \right\} \quad (41)$$

which shows that if we calculate  $C_0$  with the value of  $M$  proper to the frequency of measurement there is a further frequency correction due not only to residual inductance in  $R$ , but also due to the change of the effective resistance of  $P$  and to the change of impurity with frequency. The latter correction is less the greater the value of  $P_0$ , so that different values of  $C_0$  may be obtained if measurements are made with different

values of  $P_0$  keeping  $P_0R$  constant. Even if the values of  $C_0$  are consistent with various values of  $P_0$  this is no proof that  $r$  and  $\sigma$  are negligible, but merely that they are of the same sign and that  $r/L$  and  $\sigma/M$  are approximately equal.

Since  $r$  and  $\sigma$  normally vary as the square of the frequency it is probably preferable to write (41).

$$C = \frac{M_0}{P_0 R} (1 + \gamma \omega^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

in which  $\gamma \omega^2$  is the frequency correction peculiar to the particular arrangement of capacity inductance bridge used, and  $M_0$  is the geometrical mutual inductance.

It is possible by means of Anderson's bridge to determine the true variation of the capacity of a condenser with frequency so that the value of  $\gamma$  may be determined.

Further, by taking observations with different values of  $P_0$  keeping  $P_0R$  constant, the first correction term in (41) may be separated, and therefore since the residual inductance  $l$  of  $R$  may be measured we can determine by means of the capacity balance of the Carey-Foster bridge the true variation of  $M$  with frequency and the value of  $r - \sigma L/M$ .

Turning now to the second equation of balance (39) the resistance  $S$  includes the series resistance, ( $s$ ) representing loss in the condenser, so that putting  $S = S_0 + s$  equation (39) may be written

$$s - \sigma / \omega^2 MC = R \left( \frac{L}{M} - 1 \right) + \frac{Pl}{M} - S_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Now  $L/M$  may vary with the frequency, but otherwise the quantities on the right-hand side of (43) may be regarded as known.

$$\text{Writing } L/M = L_0/M_0 + \mu \omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$\text{and} \quad s_0 = R(L_0/M_0 - 1) + \frac{Pl}{M} - S_0, \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$s_0$  being capable of being precisely determined, we have

$$s - \sigma / \omega^2 MC - \mu \omega^2 R = s_0. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Thus the second equation in the Carey-Foster bridge balance merely gives a relation between three unknown quantities.

However, by taking observations with different values of  $R$  keeping  $PR$  constant, the value of  $\mu$  may be determined.





condition that the vector impedance of this arm shall be zero. Thus after transformation (A) the circuits are as shown in Fig. 8a in which the vector impedances are

$$\begin{aligned} \alpha &= R_1 - \sigma + j\omega(L_1 - M), & \beta &= R_2 - \sigma + j\omega(L_2 - M), \\ \gamma &= -j/\omega C', & \delta &= \sigma + s + j(\omega M - 1/\omega C). \end{aligned}$$

Applying transformation (B) to the mesh  $\alpha, \beta, \gamma$  the arm common to the generating and detecting circuits will then have vector impedance  $\alpha\beta/(\alpha+\beta+\gamma) + \delta$ , and the condition of balance is that

$$\alpha\beta/(\alpha+\beta+\gamma) + \delta = 0. \quad (48)$$

Now for small values of  $\sigma$  and  $s$ ,  $\delta$  will be small compared with  $\alpha$  and  $\beta$  in the neighbourhood of balance. Hence  $\gamma$

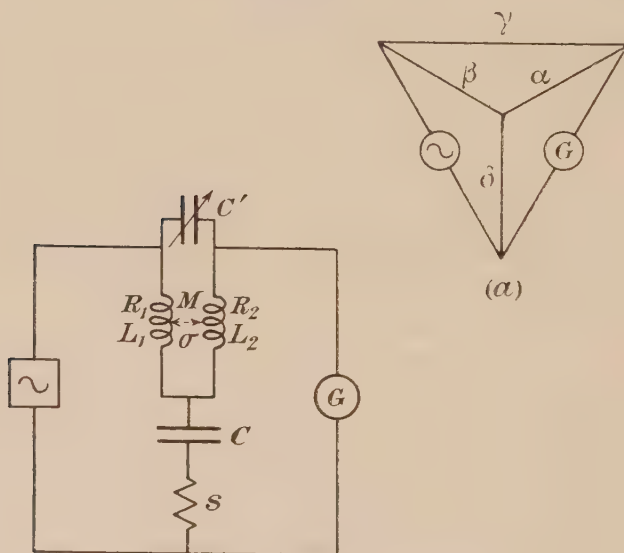


FIG. 8.—MODIFIED CAMPBELL FREQUENCY BRIDGE.

must be large, so that to the first order of small quantities (48) becomes

$$\alpha\beta/\gamma + \delta = 0. \quad (49)$$

Inserting the values of  $\alpha, \beta, \gamma, \delta$  the two conditions are

$$\omega^2 MC = 1 - \omega^2 CC' \{R_1 R_2 - \omega^2 (L_1 - M)(L_2 - M)\} \quad (50)$$

$$\sigma + s = \omega^2 C' \{R_1 (L_2 - M) + R_2 (L_1 - M)\}, \quad (51)$$

$\sigma$  being ignored where it is multiplied by the small quantity  $C'$ .

In regard to equation (50) it should be noted that if the relation

$$R_1 R_2 = \omega^2 (L_1 - M)(L_2 - M). \quad (52)$$

is satisfied then to the first order of small quantities, the original condition

$$\omega^2 MC = 1$$

is the condition for this balance. At the usual frequencies for which the bridge is suitable (1,000 cycles per second upwards) the relation (52) is practically  $L_1$  or  $L_2 = M$ . When this condition is satisfied the balances do not react on each other if adjustments are made by  $C$  and  $C'$ . Otherwise the adjustments may be made independent by inserting a small adjustable resistance ( $=S$ , say) in series with the condenser  $C$ , and adjusting by  $C$  and  $S$ .

As regards equation (51) it is not necessary to know the separate values of  $R_1$ ,  $R_2$ , &c., if the following procedure is adopted. Using the adjustments  $C$  and  $S$  repeat the balances with different values of  $C'$ . Then since  $\sigma + s + S$  is proportional to  $C'$ , the latter when plotted against  $S$  will give a straight line cutting the negative side of the axis of  $S$  at the value  $\sigma + s$ . This method may be used if  $C'$  is so large that the approximation used in obtaining equations (50) and (51) is no longer valid. For then by expansion of  $\alpha\beta'(a+\beta+\gamma)$  in powers of  $1/\gamma$  and extracting the real part we obtain instead of (51) an expression of the form

$$\sigma + s + S = AC' + BC'^2 + \dots \quad (53)$$

By using a sufficient number of values of  $C'$  the quantities  $\sigma + s$ ,  $A$ ,  $B \dots$  may be found. The same procedure holds in regard to (50), the equation being of the form

$$\omega^2 MC = 1 + A'C' + B'C'^2 + \dots \quad (54)$$

The modified frequency bridge will therefore yield the values of  $\sigma + s$  and  $MC$  at a known frequency.

#### 24. *Determination of Impurity of a Mutual Inductance and Series Resistance of a Condenser by two sets of Bridge Measurements.*

It has been shown in section 22 that the value of  $s - \sigma/\omega^2 MC$  may be determined by means of the Carey-Foster bridge, and in section 23 that the value of  $\sigma + s$  may be found by means of the modified Campbell frequency bridge if the frequency is such that  $\omega^2 MC \doteq 1$ . Hence the two sets of determinations yield  $s - \sigma$  and  $s + \sigma$  respectively at this frequency and enable

$s$  and  $\sigma$  to be determined separately. An essential point in this and similar methods is that the common point of the mutual inductance shall be the same in both bridges, as a change in common point will alter the intercapacity relations and, therefore, alter  $\sigma$ .

25. *Measurement of Impurity and Effective Resistance by Combined Observations on the Heaviside Bridge and the Carey-Foster Bridge.*

If observations of the capacity balance are taken on the Carey-Foster bridge with different values of  $P$ , keeping  $PR$  constant, the results enable  $(r - \sigma L/M)$  to be determined, where  $r$  is the added resistance in the arm  $P$  due to the alterations of the current. Again, from the Heaviside bridge, if  $r'$  is the added resistance of another inductive coil (the coil  $L$  of Fig. 6), the measurements give  $r' - 2\sigma$ . Hence, if two inductometers are used we can measure  $r - \sigma L/M$ ,  $r' - \sigma' L'/M'$ ,  $r' - 2\sigma$ ,  $r - 2\sigma'$ , the accented letters referring to the second inductometer. These four relations will, therefore, give the required quantities  $r$ ,  $r'$ ,  $\sigma$ ,  $\sigma'$ .

26. *Experimental Study of the Frequency Variations of a Mutual Inductometer.*

The inductometer studied was one of the Campbell type\* having a range of 10 millihenries sub-divided by tappings in steps of 0.1 millihenry for coarse adjustment, while fine adjustments were secured by a moving coil having a range somewhat greater than 0.1 millihenry. The scale of the moving coil was capable of being read to 0.1 microhenry, so that an accuracy of setting of one part in 100,000 of the whole range could be secured.

The method of sub-division of the fixed coils is important. This is obtained by twisting 10 equal insulated strands together, connecting in series through tappings, winding as one coil (the primary coil) and adjusting the *whole* to have a mutual inductance of 10 millihenries with regard to the fixed secondary coil. By this method the sub-division into millihenries is found to be correct to a fraction of a microhenry.

This mode of sub-division, however, introduces large capacity effects in the coil because the first and last sections are very close together. When one section only is used, the presence of the remaining nine sections, almost perfectly

\* A. Campbell, Proc. Phys. Soc., Vol. XXI., p. 69 (1908).

coupled, causes the system to behave as a 9 to 1 transformer with a capacity load, and thus introduces a large effective self capacity in the coil in use.

## 27.

The frequency variations to be determined are those of the mutual inductance and the self inductance of the secondary coil, the impurity, and the effective resistance of the secondary coil. These quantities are not usually required for the primary coil as this coil is in the source arm in alternating current bridges. The methods to be given are independent of any assumption in regard to the perfection of inductive coils and condensers. All that is assumed is that "non-inductive" resistances and the residual inductances of these resistances are invariable with frequency.

28. *Source Employed.*

The source employed was the simple valve circuit, shown in Fig. 9.

The filament current of the valve was about 4 amperes, and the filament voltage 12 volts, while the anode voltage was 400

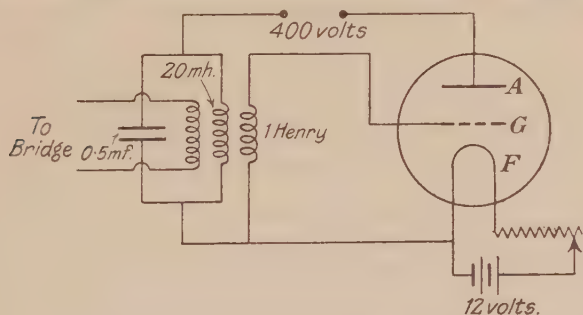


FIG. 9.—SOURCE OF ALTERNATING CURRENT.

volts. The anode inductance was 20 millihenries, and the anode capacity was adjustable continuously to 5 mfd., so that frequency could be varied from 500 ~ per second upwards. The grid inductance was about 1 henry and was not usually tuned.

A number of experiments were made using the modified Campbell frequency bridge, partly to test the modification, but mainly to form some notion of the steadiness of the valve source and its reaction to various factors.



With a frequency bridge for which  $M \doteq 10$  millihenries,  $C = 0.5$  mfd., so that the frequency was about 1,600~ per second, it was found that by placing an air condenser ( $c$ ) in parallel with  $C$ , and making the final adjustments by this condenser, the sensitivity with a 60 ohms Brown telephone was such as to enable a change of  $1 \mu\mu F$  to be detected. This means that the arrangement is capable of holding a frequency constant to one part in a million if the other elements of the bridge can be kept constant to this accuracy.

The following observations illustrate a test of the drift in frequency which occurred during a three hours' continuous run of the source, the bridge being kept on continuously :—

Time. h. m.	Temperatures		$M$ Setting $\mu H.$	Readings. $\mu\mu F.$					
	Source Table	Bridge Table.							
	°C.	°C.							
5 15	18.8	20.5	...	...					Valve on Bridge on minute readings. $\frac{1}{2}$ minute readings
5 30	...	...	...	...					
5 43	...	...	10010.0	{ 129 135 141 147 159					
5 52	...	...		{ 165 170 178 185 190					
6 15	...	...	10012.0	{ 132 134 136 138 139.5					
6 20	18.7	20.5		{ 142 143.5 146 148 150					
6 47	...	...	10013.0	{ 124 125 127 128 129					
6 52	18.8	20.5		{ 129.5 131 131.5 134 134					
7 15	...	...	10013.0	{ 166.5 167 167.5 167.5 168					
7 20	18.7	20.5		{ 168.5 169 170 170.5 171					
7 45	...	...	10013.0	{ 187.5 188 188 189 189.5					"
7 50	18.65	20.5		{ 189 189 189.5 190.5 190					
8 13	...	...	10013.0	{ 197 197 196 196.5 196					
8 18	18.6	20.5		{ 196.5 196.5 196.5 197 197					

From the groups of observations of  $c$  the values of the inductance  $M$  which would have secured a balance without change of  $c$  were deduced for the moments given in the following table, which also gives the drifts at these moments expressed in terms of  $M$ . Since for the frequency bridge  $\omega^2 MC = 1$ , the changes in adjustments of  $M$  with time are proportional to  $1/\omega^2$ .

Time. h. m.	Corrected $M$ . $\mu H.$	Drift per min.
5 45	10011.42 <sub>1</sub>	0.071
6 15	13.32 <sub>0</sub>	0.040
6 45	14.20 <sub>0</sub>	0.022
7 15	14.66 <sub>2</sub>	0.010
7 45	14.87 <sub>9</sub>	0.004 <sub>8</sub>
8 15	14.96 <sub>8</sub>	0.000 <sub>8</sub>

The results are plotted in Fig 10, the tangential lines representing the observed drifts. The curve *B* shows the observed drift on a Carey-Foster bridge under similar conditions. The balance for this bridge is practically independent of frequency, so that this curve gives some indication as to what portion of the drift is contributed by the elements of the bridge.

The curve shows the extremely regular behaviour of a valve source when working under favourable conditions.

The whole change in  $2\frac{1}{2}$  hours is only 2 parts in 10,000 when expressed in terms of frequency, the change in the last hour

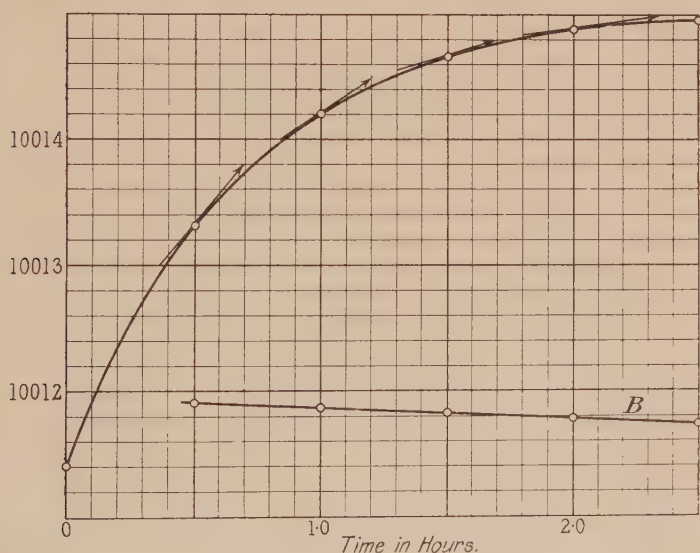


FIG. 10.—DRIFT IN FREQUENCY OF A VALVE SOURCE AS INDICATED BY THE MODIFIED FREQUENCY BRIDGE.

being 15 in a million, and during the whole period there was no detectable departure from a regular drift. Other tests were made showing that the source was susceptible to various factors. A change of valve condenser after the circuit had been brought to a steady state immediately produced a further drift in frequency towards a new state of equilibrium (apart altogether from that due to any change in the value of the capacity). Some condensers produced a drift of opposite sign.

An increase in condenser current of 2 per cent. produced an

increase in frequency of two parts in 100,000, and a reduced filament current also increased the frequency by a few parts in 100,000, these changes taking place immediately. By running the valve on full filament current for some time it was found possible to bring about a steady state for a reduced filament current much earlier than is indicated by Fig. 10. The source frequency is also susceptible to variations in load on the bridge, the change of frequency being such as would be expected from the reduced effective inductance of the valve circuit when coupled to the bridge circuit. It is therefore necessary when bridge measurements are being taken which involve extreme steadiness of frequency and also considerable changes in the effective resistance of the bridge to have an auxiliary frequency bridge to hold the frequency constant. The modified frequency bridge has been found very useful for this purpose.

### 29. *Earth Capacities.*

It was found to be necessary to make corrections for earth capacities in certain of the bridge measurements. In bridges of the Wheatstone bridge type (the Heaviside bridge and the Carey-Foster bridge) the Wagner earthing device may be used,\* but actually a method of correction was employed involving two measurements with different earth points. This method will form the subject of a separate Paper.

In the case of the frequency bridge, the bridge was connected to earth at the point common to the source, detector and condenser arm as the earth capacities (assumed to be condenser arms connecting the branch points to earth) will either have no effect or act as an added capacity to  $C$  (Fig. 8). As this bridge was simply used to measure *changes* in frequency or to determine quantities in which  $C$  was involved, only to the first order of small quantities, no appreciable error could arise from this cause.

### 30. *Frequency Coefficient of Mutual Inductance.*

For this purpose a Carey-Foster bridge method was employed, using a condenser of nominal value 0.5 mfd., whose variation with frequency had been determined by Anderson's bridge. The curve of variation for this condenser is shown in Fig. 11 in which the variations  $\Delta C$  in micro-microfarads are plotted against frequency. Since the leads to the condenser and

\* See Laws, "Electrical Measurements."

the resistance  $S$  are included in the condenser arm (Fig. 7) the effect of their residual inductances must be included in the value of  $C$  in equation (41). Let the residual inductance of the condenser arm be  $\lambda$ . Then if  $C$  in (14) is replaced by  $C(1+\omega^2\lambda C)$  the equation becomes

$$C = \frac{M}{P_0 R} \left\{ 1 - \frac{r - \sigma L/M}{P_0} + \frac{\omega^2 l(L-M)}{P_0 R} - \omega^2 \lambda C \right\}. \quad (55)$$

Writing  $M = M_0(1 + \gamma'\omega^2)$ , (55) becomes

$$C = \frac{M_0}{P_0 R} (1 + \gamma\omega^2), \quad (56)$$

in which

$$\gamma\omega^2 = \gamma'\omega^2 - \frac{r - \sigma L/M}{P_0} + \frac{\omega^2 l(L-M)}{P_0 R} - \omega^2 \lambda C, \quad (57)$$

and is the frequency correction of the whole bridge outside the condenser.

$\gamma$  may be determined by the variations in the  $M$  settings

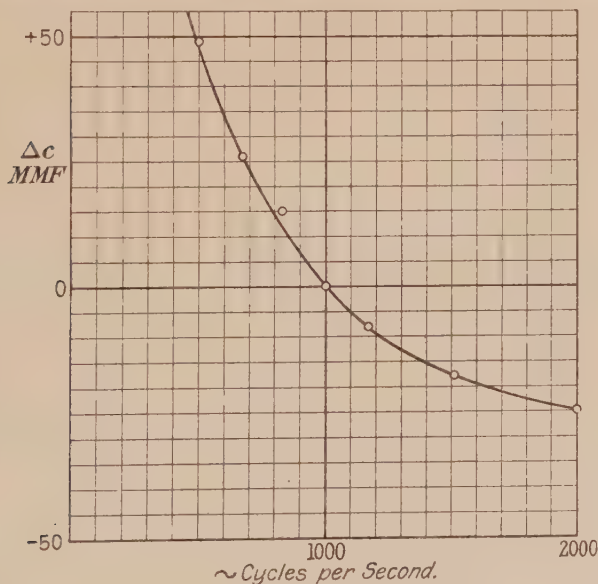


FIG. 11.—CHANGE OF CAPACITY OF  $\frac{1}{2} \mu F$  CONDENSER WITH FREQUENCY.

required to balance the known variations in  $C$  at different frequencies. Actually coarse adjustments only were made by the moving coil of  $M$ , the fine adjustments being made by





when the values of  $s_0$  are determined with different values of  $R$ , keeping  $P_0R$  constant, the values of  $a$  and  $b$  being by (46)

$$a = s - \sigma / \omega^2 MC,$$

$$b = \mu \omega^2,$$

$s$  being the true series resistance of the condenser  $C$ ,  $\sigma$  the impurity of the mutual inductance  $M$ , and  $\mu \omega^2$  the frequency correction of  $LM$ . The values of  $a$  and  $b$  were determined using the condenser already employed in section 30 for three different frequencies. The results are given in the following table, the columns marked " $s_0$  calculated" being obtained from (58), using the best values of  $a$  and  $b$  for the observations.

TABLE.

Carey-Foster Bridge.

Variation of  $P$  and  $R$ .

$$M_0 \doteq 10 \text{ mh.}$$

$$L_0 = 26.110 \text{ mh.}$$

$$C \doteq 0.5 \text{ } \mu\text{F.}$$

$$PR = 20,000.$$

$f$  = frequency  $\sim$  per second.

$R$ ohms.	$f=1,000.$		$f=1,600.$		$f=2,000.$	
	$s_0$ observed ohms.	$s_0$ calculated.	$s_0$ observed ohms.	$s_0$ calculated.	$s_0$ observed ohms.	$s_0$ calculated.
50	0.098	0.098	0.096	0.096	0.128	0.130
100	0.128	0.127	0.167	0.165	0.143	0.241
200	0.185	0.185	0.302	0.303	0.464	0.462
$a =$	0.069		0.027		0.019	
$-b =$	0.00058		0.00138		0.00222	
$-\frac{b}{\omega^2} =$	$1.42 \times 10^{-11}$		$1.38 \times 10^{-11}$		$1.40 \times 10^{-11}$	

Hence, as a mean value

$$\mu = -1.40 \times 10^{-11}.$$

A check on this determination may be obtained by combining the results of observations on the Carey-Foster bridge with observations on Anderson's bridge.\*

For the Carey-Foster bridge we have equation (46), viz.,

$$s_0 = s - \sigma / \omega^2 MC - \mu \omega^2 R, \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and for Anderson's bridge,\* if  $s'$  is the series resistance used with the condenser,

$$s' = A - s,$$

\* The use of Anderson's bridge for determining the true variation of the series resistance and capacity of a condenser will form the subject of a separate Paper.

where  $A$  is a quantity independent of frequency involving the remaining residuals in Anderson's bridge.

Hence

$$s_0 + s' = A - \frac{\sigma}{\omega^2 MC} - \mu \omega^2 R,$$

or, since  $\sigma$  is proportional to  $\omega^2$

$$s_0 + s' = A' - \mu \omega^2 R, \quad . . . . . (59)$$

from which  $\mu$  may be found by observations at different frequencies.

The results obtained by this method for three different condensers are given in the following table:—

TABLE.

Anderson's Bridge and Carey-Foster Bridge.

For the Carey-Foster Bridge—

$M = 10$  mh.  $C = 0.5 \mu F$ .  $L_0 = 26.110$  mh.  $P = 400$  ohms.  $R = 50$  ohms.

$$s_0 = R \left( \frac{L_0}{M_0} - 1 \right) + \frac{Pl}{M} - S.$$

$s'$  = series resistance in Anderson's Bridge.

Frequency per second.	—	Condenser I.	Condenser II.	Condenser III.
1,000	$s_0$ ... ..	0.100	0.077	—0.016
	$s'$ ... ..	0.625	0.650	0.715
	$s_0 + s'$ { observed calculated	0.725	0.727	0.699
		0.726	0.726	0.700
1,600	$s_0$ ... ..	0.096	0.072	+0.001
	$s'$ ... ..	0.675	0.700	0.740
	$s_0 + s'$ { observed calculated	0.771	0.772	0.741
		0.771	0.771	0.745
2,000	$s_0$ ... ..	0.127	0.096	0.042
	$s'$ ... ..	0.685	0.715	0.751
	$s_0 + s'$ { observed calculated	0.812	0.811	0.793
		0.812	0.812	0.786

The calculated values of  $s_0 + s'$  were obtained from formula (59), using as the values  $A'$  for the three condensers.

$$A' = 0.698 \quad 0.698 \quad 0.672,$$

and for  $\mu$  the common value  $-1.425 \times 10^{-11}$ , which are the values agreeing best with the observations. This value of  $\mu$  differs from the previous determination by 2 per cent.

As regards  $A'$ , the capacity of condenser III. was a few per

cent. higher than those of condensers I. and II., which differed only by 2 or 3 parts in 10,000. The differences in  $A'$  are sufficiently accounted for by the changes in the residuals in Anderson's bridge brought about by rebalancing for condenser III.

### *Determination of $\sigma$ .*

Using the known value of  $\mu$  in equation (46) the values of  $s - \sigma \omega^2 MC$  at a frequency of 1,600 cycles per second for the three condensers are

$$+0.025, \quad +0.001, \quad -0.070.$$

Since  $\omega^2 MC = \frac{1}{2}$  at this frequency, these are the values of  $s - 2\sigma$ .

Another relation between  $s$  and  $\sigma$  can be found by the modified frequency bridge (section 23). For this bridge  $\omega^2 MC \doteq 1$ , and this condition may be secured by paralleling two of the condensers, say, I. and II.

By the usual laws of combination, and by (53) the modified frequency bridge measures

$$(s_1 + s_2)/4 + \sigma.$$

The value obtained with Condensers I. and II. was

$$(s_1 + s_2)/4 + \sigma = 0.096.$$

Now,

$$s_1 - 2\sigma = 0.025$$

$$s_2 - 2\sigma = 0.001,$$

so that, solving for  $s_1$ ,  $s_2$ ,  $\sigma$ ,  $s_1 = 0.115$ ,  $s_2 = 0.091$ ,  $\sigma = 0.045$  ohms. Using this value of  $\sigma$  for the Condenser III.,

$$s_3 = 0.020.$$

Since  $\sigma$  varies as the square of the frequency, the value of  $\sigma$  for the inductometer studied is 0.017<sub>5</sub> ohm, at a frequency of 1,000 cycles per second.

### *32. Summary of Inductometer Corrections.*

From the results obtained in sections 30 and 31 the corrections for frequency for the inductometer studied are as follows:

At a frequency of 1,000  $\sim$  per second the mutual inductance



has a positive correction of 9.8 parts in 10,000,\* the self-inductance of the secondary coil has a positive correction of 7.6 parts in 10,000, the impurity is 0.017<sub>5</sub> ohm, and the effective resistance of the secondary coil is greater than its direct current value by 0.042 ohm.

All the corrections vary as the square of the frequency.

These corrections only hold when the mutual inductance has a common point such that when the coils are in series, the mutual inductance opposes the self-inductances.

### 33. *Determination of Impurity by Measurements of Effective Resistance.*

If an inductive coil  $L$  is placed in series with a condenser  $C$ , non-inductive resistance and an alternating-current ammeter so as to form a simple circuit, the current induced in the circuit by an E.M.F.  $E$ , alternating with frequency  $\omega/2\pi$ , is  $I = E / \{R^2 + (\omega L - 1/\omega C)^2\}^{\frac{1}{2}}$ , so that if  $C$  is adjusted to resonance with  $L$ , the current is a maximum and has the value  $I_{\max.} = E/R$ .

By observing the maximum current with different values of the non-inductive resistance the resistance of the remainder of the circuit may be deduced. The method therefore gives the resistance of the coil and condenser in series. Assuming the condenser to have the same power-factor at all frequencies, then remembering that the value of the capacity is altered for different frequencies the series resistance of the condenser will increase as the frequency, while if the coil is wound with thin wire the resistance of the coil increases as the square of the frequency, so that the losses in the condenser will become less and less important as the frequency increases. Moreover, at the higher frequencies the resonating capacity may be made so small that an air condenser may be used and thus the condenser losses may be still further reduced.

In order to determine the impurity of a mutual inductance by this method, the two coils are connected at one point ( $B$  in Fig. 7A) and the effective resistances measured across the points  $AB$ ,  $BC$ ,  $CA$ . If at a particular frequency the values of the effective resistances are  $R_1$ ,  $R_2$ ,  $R_3$  respectively, then since  $R_3 = R_1 + R_2 - 2\sigma$  we have  $\sigma = \frac{1}{2}(R_1 + R_2 - R_3)$ .

The following values were obtained for a fixed mutual in-

\* This result is in good agreement with an earlier determination on the same inductometer by Campbell. (Proc. Roy. Soc., Vol. LXXXVII., 1912, p. 406.)

ductance of 10 millihenries, wound on marble with No. 26 d.s.c. wire.

Primary coil  $L=21.9$  millihenries. D.C.R.=12.54 ohms.

Secondary coil  $L=21.5$  millihenries. D.C.R.=12.25 ohms.

*Alternating Current Resistances.*

Frequency. ~ per second.	Primary.	Secondary. ohms.	Series opposition.
17,300	36	34	59
25,000	62	59	99
33,300	105	98	154

The resistances are plotted against the square of the fre-

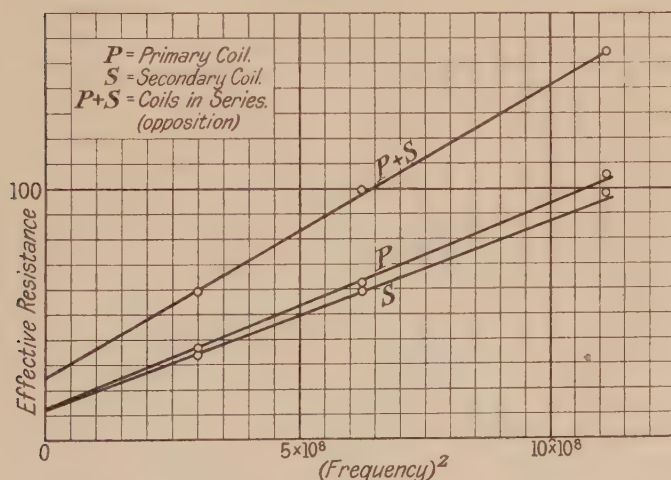


FIG. 12.—EFFECTIVE RESISTANCE OF COILS OF INDUCTOMETER.

quency in Fig. 12, and from the mean straight lines we find at  $f=1,000$  ~ per second, the following corrections :—

Primary	0.081 ohm,
Secondary	0.075 ohm,
Series opposition	0.118 ohm,

from which the impurity is 0.020 ohm.

The same inductance was tested at a frequency of 1,600 ~ per second upon the modified frequency bridge, using a  $1\mu F$  condenser of known power-factor. The deduced impurity at

a frequency of 1,000~ per second by this method was 0.0207 ohm.

### 34. *Errors in Power Factor Measurements due to Imperfections of Mutual Inductance.*

Equation (46) may be written

$$\omega s_0 C = \omega s C - \sigma / \omega M - \mu \omega^3 M / P$$

and since  $\omega s C$  is the power factor of the condenser,  $\omega s_0 C$  will be apparent power-factor of the condenser as deduced from Carey-Foster bridge observations upon the assumption of a perfect mutual inductance. It is seen that the apparent power-factor is subject to two errors, one due to impurity and proportional to the frequency, and the other due to the change of  $L/M$  with frequency, and proportional to the cube of the frequency. The latter error depends also on the resistances employed in the bridge. For the inductometer studied, the error due to impurity is  $2.8/10^4$  at a frequency of 1,000, and tends to make the apparent power-factor low, while the error due to the change of  $L/M$  is  $0.9/10^4$  (when  $P=400$  ohms), tending to make the apparent power-factor too high. These values hold whatever capacity is being measured. Since the power-factor of a good condenser is of the order of a few parts in 10,000, it is clear that the Carey-Foster bridge measures the imperfections of its own mutual inductance, rather than the power-factor of the condenser at the higher telephonic frequencies. However, by a proper use of intercapacity the imperfections of the inductance may be compensated, the compensation holding over the whole range of frequencies for which the eddy current and capacity effects vary as the square of the frequency. A method of compensating a mutual inductance for change of mutual inductance with frequency and for impurity has already been given in section 19. This compensation is suitable for a mutual inductance used for the measurement of self-inductance upon a Heaviside bridge, but when using the inductance upon a Carey-Foster bridge for power-factor measurements, what is required is that the impurity shall be zero, and the variation of  $L/M$  shall be zero.

Let a condenser  $C'$  connect the open ends of the inductometer. Then by equations (8), (9), (11), the change in the frequency coefficient of  $L_2/M$  is  $\omega^2 C' (L_2 - M)(L_1 L_2 - M)^2 / M L_2$  and the change in  $\sigma$  is

$$-\omega^2 C' \{R_2(L_1 - M) + R_1(L_2 - M)\}.$$

For the inductometer studied the values of  $L_1$ ,  $L_2$ ,  $R_1$ ,  $R_2$ ,  $M$  were

$L_1 \doteq 7$  millihenries,  $L_2 \doteq 26$  millihenries,  $M = 10$  millihenries,  
 $R_1 \doteq 10$  ohms,  $R_2 \doteq 18$  ohms.

Using these values we find that if  $C' = 1,100 \mu\mu F$  the change of  $L_2/M$  due to  $C'$  is  $+2.2/10^4$  at a frequency of 1,000  $\sim$  per second, and this will just balance the existing variation of  $-2.2/10^4$ . Further with this value of  $C'$  a negative impurity of amount 0.0175 ohm will be introduced if  $R_1$  is made equal to 21.5 ohms, so that by adding 11.5 ohms to the primary coil and using an added intercapacity of 1,100  $\mu\mu F$  across the whole combination the corrections for this inductometer will be automatically compensated.

The experimental work was carried out at the National Physical Laboratory. I wish to thank a number of my colleagues, particularly Mr. D. W. Dye, for valuable suggestions during the course of the experimental investigation.

#### DISCUSSION.

Dr. A. RUSSELL said he presumed the Paper was on an experimental basis, as the author had mixed theory and experimental approximation throughout, *e.g.*, in treating the variation of mutual and self-inductance due to what he terms the mutual capacity and self-capacity of the two coils. He thought G. A. Campbell's method was probably not good for high frequencies. The author neglected skin effects. He supposed he was familiar with Curtis's Paper on skin effects in parallel wires in which formulæ are given for the eddy-current losses which are fully verified by experiment. Did these results agree with Mr. Butterworth's? He fully agreed with the author's remarks on the Carey Foster Bridge. In the equation for  $\omega^2 C$  the equating of the last factor to zero is the condition that the phase difference between the two currents is a maximum.

Dr. D. OWEN said the Paper should prove of great value in inductance measurements of high precision. The first impression on reading it might perhaps be that alternate-current methods introduced complications; but this was really a measure of the great accuracy which these methods of measurement afforded. The Paper showed that many conditions have to be borne in mind in obtaining definite results; one would like to know whether the effects of capacities to earth had also been kept in view.

Mr. ALBERT CAMPBELL (communicated remarks): Mr. Butterworth's elaborate and thorough research will be of great value to all who are interested in precision measurements in which inductometers are used. The effects which he has investigated in general only give trouble in the rather extreme cases, *e.g.*, at the higher audio-frequencies, or when extremely minute power losses are being measured (in condensers of low power factor). In such cases it is probably better to use a more elaborate form of inductometer, wound with multiple pancake coils or otherwise so as to minimise the capacitance effects. Such construction would, of course, increase the volume of the instrument and add to its cost. The inductometer which Mr. Butterworth has studied is of early type, in which the higher subdivisions are all obtained by stranding the wires. In the more modern inductometers of this kind the higher coils are not stranded together; each subdivision consists of an individual coil separately adjusted. Stranding is

still employed for the lower sections where the capacitance effects are of much less relative importance. The methods described by the author for applying automatic corrections will be valuable in practice and the whole Paper contains much that may ultimately be of importance to the radio engineer.

Mr. BUTTERWORTH, in reply to Dr. Russell, said he was unaware of the criticism of G. A. Campbell's method which he had mentioned, but it was all right if the frequency was under 2,000 or so. He had seen Curtis's Paper. As far as skin effects are concerned it should be taken in conjunction with a Paper by himself (at present in the course of publication), in which a much shorter formula was obtained.

In reply to Dr. Owen the question of earth capacities is dealt with in a Paper originally included as an appendix to the present one, but to be published later as a separate Paper.



XXIX. *New Specific Heat Apparatus.* By EZER GRIFFITHS,  
D.Sc. (*The National Physical Laboratory.*)

RECEIVED APRIL 11, 1921.

ABSTRACT.

The Paper is a description of an apparatus for the determination of the specific heats of materials such as cork, charcoal, &c., which are difficult to deal with by the ordinary methods. The calorimeter is a closed cylinder rotating on a horizontal axis. The heating coil and the thermo-elements project into the calorimeter and are maintained stationary. The material under test fills about one-third of the calorimeter and is mixed by being carried round by the fins attached to the internal surface. To ensure equalisation of the temperature, the material is ground to a state of fine sub-division before test.

By reducing the weights of all parts to a minimum a calorimeter of 4 litres capacity has been constructed whose water equivalent is less than 100 grams. The jacket surrounding the calorimeter is made of copper wound with a heating coil. In an experiment the temperature of this jacket is maintained equal to that of the calorimeter, thus eliminating the corrections for heat loss by radiation and convection.

THE usual "method of mixtures" for specific heat determinations is simple, and gives satisfactory results when the test specimens happen to take the form of metallic blocks. But when materials such as cork, chocolate, concrete, &c., have to be examined, serious experimental difficulties are encountered when an attempt is made to use the above method, so it is better to design an apparatus specially adapted to the work in hand. The apparatus described below was constructed for the study of the thermal capacity of light, bulky materials.

Cork, charcoal, slag wool, and diatomaceous earth are materials extensively used for cold storage insulation, and it is sometimes necessary to make calculations as to the penetration of heat through a wall when the temperature of one of its faces is suddenly raised.

In the course of an investigation on heat insulators for the Food Investigation Board, it became necessary to determine the specific heats of various heat insulating materials.

An electrical method was employed, because it was desirable to obtain the specific heat over a narrow range of temperature. Heating materials, such as charcoal, to 100 deg. alters the moisture content of the samples, and some preliminary tests proved that the charcoal and diatomaceous earth on heating to 105°C. showed a loss of weight of 6.9 and 8.8 per cent. respectively, due to the water driven off. Hence, any method

which necessitated the heating of the sample through a wide interval of temperature was inadmissible.

### *Description of Apparatus.*

The calorimeter (Fig. 1) was a large cylindrical vessel of thin sheet aluminium. It was carried on a horizontal axis, and rotated at a uniform rate by an electric motor through a gear box. The usual speed of rotation was about one revolution in two seconds.

The material, in the state of fine sub-division, occupied about one-third to one-half the volume of the calorimeter. Helical fins were riveted to the internal surface, which, by the rotation of the cylinder, carried the contents around, and produced a very thorough mixing up of the powder.

A heating coil projected into the cylinder, and it was held stationary. The watts dissipated in this coil were measured by observations of the current and the potential difference at its ends.

Temperature was measured by six thermo-junctions of iron-constantan wire suitably supported so as to give the temperature at various points, but the observed differences were found to be negligibly small.

The rotation of the cylinder carried the material up, and, on reaching various heights, it dropped off the fins and fell through the heating coil on to the thermo-junctions. The helical fins were so arranged that half the set tended to carry the material in one direction by screw motion, and the other half in the opposite direction, the object being to produce mixing of the contents in the longitudinal direction. It was deemed advisable to assist the mixing by arranging the apparatus so that it would be swayed about an axis at right angles to that of the rotation, the effect of this being to throw the material from end to end of the cylinder.

By reducing the weights of the various parts to a minimum, it has been possible to construct the apparatus with a heat capacity of only 103 gram calories, whilst the calorimeter has a cubical capacity of 4,300 c.c.s. A calorimeter of this volume with such a small thermal capacity would have a relatively big heat loss by radiation and convection, if it were allowed to radiate to constant temperature surroundings. This correction was eliminated by arranging the rotating calorimeter within a large copper drum wound with a heating coil. During the experiment the surroundings were heated up at the same

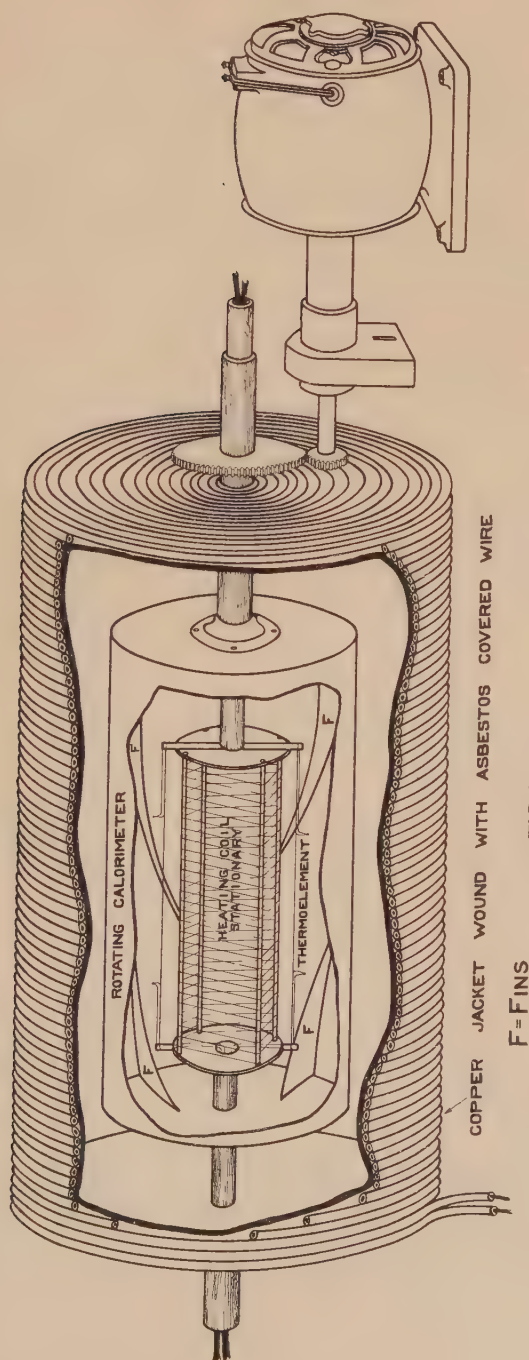


FIG. 1.

rate as the calorimeter by suitably controlling the current through the coil around the drum.

Hence, during the actual experiment, the material under test was heated from one nearly stationary temperature to another, and no account had to be taken of any heat losses.

The heating coil on the drum was made of asbestos-covered resistance wire, the ends as well as the cylindrical portions of the drums being heated. The temperature of the internal face was obtained by means of thermo-junctions.

### *Preparation of the Materials for Test.*

These materials in their natural state are either granular or of fibrous structure. Consequently it was necessary to reduce them to a fine state of subdivision to ensure that the mixing up during the stirring produced uniformity of temperature throughout the mass. The charcoal and slag wool were ground up in an iron mortar and the iron abraded from the mortar was removed by the aid of a magnet. Cork was somewhat more difficult, but it was found that a meat-mincer effectively reduced it to a state of suitable subdivision. The operation of grinding had to be carried on slowly, otherwise the heat generated was considerable; it was, in fact, found that too vigorous operation caused charring of the cork. The moisture content of the samples was determined during the course of the work.

The procedure in carrying through a test was as follows:—

A quantity of the material was ground to an exceedingly fine powder, weighed and inserted into the calorimeter. This was set in rotation, and after an interval of time the temperature would attain a steady state. The electric supply was then switched on to the heating coil and a little previously to the jacket, the appropriate values of the heating currents having been determined from preliminary experiments. The temperatures of both the calorimeter and the jacket then rose at a steady rate, and adjustments of the supply to the jacket were made to keep them approximately equal. After the temperature had risen through about 15 deg., the supply was switched off and observations of the temperature of the calorimeter continued until the steady state had been reached. From a knowledge of the initial and final temperatures as well as of the total watts supplied during the run, the specific heat of the material

is readily calculated. Experience showed that, owing to the fact that the response of the jacket temperature to the heating coil was less rapid than that of the calorimeter, it was necessary to switch on and cut down the supply to the jacket about two minutes before switching on and off the supply to the calorimeter.

The result would be that the rise and the final steady temperature of the jacket would correspond very closely with that of the calorimeter. It was, of course, necessary to

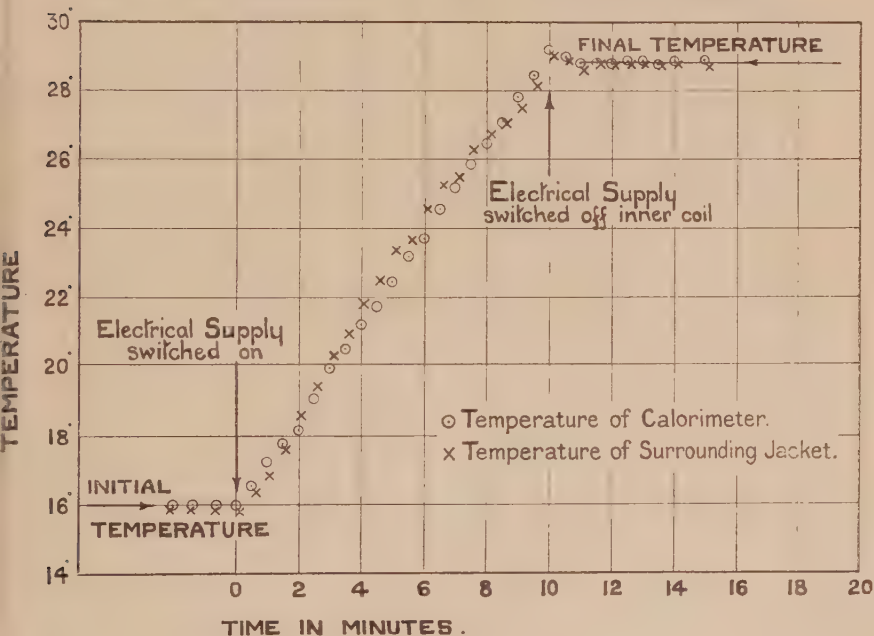


FIG. 2.

maintain a small current through the coil on the jacket to compensate for the heat lost from its surface at the higher temperatures. Tests and calculations were made to determine the small correction for the heating of the charge by the stirring.

Correction was also made for the heat absorbed by the air in the space unoccupied by the charge in the calorimeter. The values of the specific heats given correspond to the material in the closely packed state and, of course, include that of a little air.



A typical series of observations are shown in Fig. 2.

*Specific Heat Results.*  
(At mean temperature of 25°C.)

Material.	Moisture Content.	Specific Heat.
Charcoal ... ..	6.89 per cent.	0.29 <sub>9</sub>
Slagwool ... ..	0.02 "	0.17 <sub>1</sub>
Diatomaceous earth ... ..	2.09 "	0.23 <sub>2</sub>
Baked cork (slab) ... ..	—	0.43 <sub>1</sub>
Granulated cork ... ..	3.45 "	0.43 <sub>8</sub>

During the course of some more recent experiments on the specific heats of materials used in cable insulation, the above apparatus was modified in several details by Mr. R. R. Strand, Senior Observer in the Heat Department.

To facilitate the adjustment of the temperature of the calorimeter to equality with the jacket differential thermo-

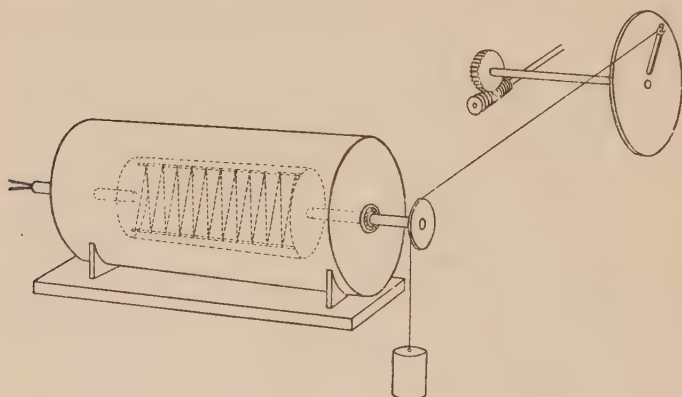


FIG. 3.

couples were attached to the surfaces of both calorimeter and jacket, and the method of stirring modified so that only a partial rotation is given to the calorimeter. The form of gearing required to effect this is shown diagrammatically in Fig. 3. The worm-driven wheel makes about one revolution in two seconds, and from the crank pin a string is led over the pulley fixed to the axle of the calorimeter. By suitably proportioning the diameter of the pulley and the radial distance of the crank pin the magnitude of the angular rotation of the calorimeter can be adjusted.

The jacket was wound with flat nichrome tape insulated

by very thin micanite so as to obtain quicker response of the temperature of the surroundings to alterations in the energy supply.

In this particular calorimeter the heating coil was made of a flat, open grid of platinum wire occupying the diametrical plane. This coil was also the thermometer.

Initially a minute current was passed through the coil and the potential difference at its ends compared with that at the terminals of a standard coil connected in series with it. This gave the initial temperature rise of the calorimeter due to the stirring alone. The connections were then switched over to the heating supply circuit and the watts dissipated in the heating coil measured in the usual manner. Finally, the connections were switched back to the thermometer circuit and observations taken until the final equilibrium temperature had been attained. The advantage of the resistance thermometer is that it gives the average temperature over the area occupied.

We wish to thank our director, Sir Joseph Petavel, for his kind interest and encouragement.

XXX. *On Encounters between Non-spherical Gaseous Molecules.* By A. O. RANKINE, D.Sc., Professor of Physics in the Imperial College of Science and Technology.

RECEIVED APRIL 14, 1921.

ABSTRACT.

This Paper is an extension of the author's previous work on molecular dimensions and structure derived from the combined data of the kinetic theory of gases and of X-ray crystal measurements (Proc. Roy. Soc., A. Vol. XCVIII, p. 360). The cases previously considered have been confined to molecules built up of atoms all of equal size. In this Paper this limitation is removed, and unequal atoms are contemplated.

The results arrived at cover all cases of diatomic molecules and certain special cases in which the molecules are polyatomic. It will be possible to test their validity when the appropriate viscosity data become available.

1. IN dealing with the kinetic theory of gases it has hitherto been the almost invariable custom to regard the molecule as spherical, although it is well recognised that, except for monatomic molecules, this is unlikely to be true. The reason for this has, no doubt, been lack of information from which to assign probable shapes to molecules consisting of more than one atom. Recently the situation in this respect has changed, and the developments of the Lewis-Langmuir\* theory of molecular constitution have provided good grounds for specifying the comparative sizes and shapes, not only of atoms themselves, but also, in many cases, of the molecules they form by combining with one another. Added to this, the dimensions in question have been reduced to a quantitative basis by W. L. Bragg's† deductions from X-ray crystal measurements, so that we are able to state with some confidence the actual distance apart of the centres of two atoms which are in chemical combination.

It has already been possible to apply this new set of data usefully to the kinetic theory. For example, a chlorine molecule, on the Lewis-Langmuir theory, is conceived to be an arrangement of electrons nearly identical in shape with that obtained by placing two argon atoms so close together that their outer electron shells are touching one another.

\* I. Langmuir, Journ. Amer. Chem. Soc., XLI, p. 868 (1919).

† W. L. Bragg, "Phil. Mag.," Vol. XL, p. 169 (1920).

The author\* has been able to show that two such atoms, with this degree of proximity, may be expected to behave, from the point of view of the kinetic theory, as a molecule of chlorine in fact does. On the same lines it has been proved that molecules of nitrous oxide and carbon dioxide both behave kinetically like hypothetical molecules each consisting of three neon atoms with their centres equally spaced on a straight line and with outer electron shells contiguous, thus affording further support to Langmuir's conclusions derived from other considerations.

In all these cases, however, the molecules dealt with have been constituted of atoms of equal size. It is the purpose of the present Paper to extend the theory so as to include molecules containing unequal atoms. Unfortunately, there exist at present no reliable viscosity data by means of which to test this new extension, so that its application must await the results of further experimental investigations.

2. According to the kinetic theory of gases the atoms of a monatomic gas behave, when in thermal agitation, like hard elastic spheres, which exert slight mutual attraction on one another and frequently collide. S. Chapman† has shown that this model of the atom, viz., a hard attracting sphere of radius  $\sigma$ , gives, for the monatomic inert gases, results which are very closely in accordance with experiment. His formula giving the magnitude of  $\sigma$  is, apart from a negligible term,

$$4\pi\sigma^2 = \frac{0.491\rho\bar{C}}{\sqrt{2}\nu\mu(1+S/T)}, \quad \dots \dots (1)$$

where  $\rho$ =gaseous density,  $\bar{C}$ =mean molecular velocity,  $\nu$ =number of molecules per unit volume,  $\mu$ =viscosity,  $S$ =Sutherland's constant, and  $T$ =absolute temperature. The quantity  $\pi\sigma^2$ , it will be noted, is equal to the area of any and every central section of the molecule, which is supposed to be, in this case, spherical.

The measurement of the viscosity of the gas, and its variation with temperature (which determines Sutherland's constant), thus enables us to evaluate  $\pi\sigma^2$  for monatomic molecules. In reality, what we find is the area which the molecule presents as a target for other molecules approaching from all directions.

\* A. O. Rankine, Proc. Roy. Soc., A., Vol. XCVIII., p. 360.

† S. Chapman, "Phil. Trans.," A., Vol. CCXVI., p. 279 (1916).

It is permissible to apply formula (1) to molecules which are not spherical provided that we interpret  $\pi\sigma^2$  as being an average area,  $\overline{A}$ , which measures the mean target area presented by the molecule for all possible orientations. But we can go no further in the investigation of the molecular dimensions until we know how the constituent atoms are arranged.

3. If, as is evident from Chapman's work, single atoms behave in collision like hard spheres of definite radius, it is reasonable to suppose that when in chemical combination they preserve this property of apparent hardness. To a first approximation, also, we may continue to regard them as spheres. No doubt there are distortions, but there is no obvious way of estimating their extent, and we will, therefore, leave them out of account at present. According to this view we may picture a molecule as an assemblage of hard spheres arranged in a definite manner, and presenting, according to aspect, different target areas. Our purpose is to calculate the mean value of this area from all possible directions. It is evident that the problem cannot be treated conveniently by means of a perfectly general case. Each alteration of the number and size of the constituent spheres, and each variation of the positions of their centres, develops a new special case which is best examined separately. I\* have already given the solution for a molecule consisting of any number of equal spheres with their centres equally spaced on a straight line, and have applied it with success to actual molecules which probably have that form. The present extension consists of examining the new case where the spheres are of unequal radii, but still have their centres disposed rectilinearly, so that there is an axis of symmetry. This arrangement is necessarily the only possible one for all of the numerous diatomic molecules, such as, for example, HCl.

4. Let us consider first of all a molecule consisting of two hard spheres of radii  $\sigma_1$  and  $\sigma_2$  ( $\sigma_1 > \sigma_2$ ), the distance between their centres being  $2d$ , as indicated in Fig. 1. The evidence so far available from X-ray crystal measurements (which enable us to specify  $O_1 O_2$ ) is that the centres of atoms in chemical compounds are so close together that the hard spheres to which the atoms correspond overlap to some

\* *Loc. cit.*



extent. In other words,  $2d$  is never greater than  $\sigma_1 + \sigma_2$ . For this reason the present calculation is limited to values of  $2d$  between  $\sigma_1 + \sigma_2$  and  $\sigma_1 - \sigma_2$ , *i.e.*, to the range between external and internal contact of the spheres.

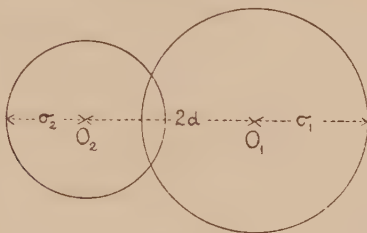


FIG. 1.

The axis  $O_1 O_2$  of the molecule may have all possible orientations distributed at random. Let Fig. 2 represent the parallel projection in a direction making an angle  $\theta$  with

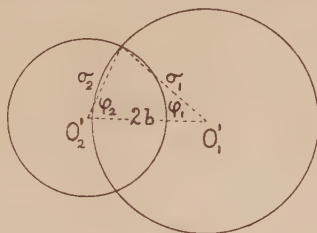


FIG. 2.

the axis.  $O_1$  and  $O_2$  are projected into  $O'_1$  and  $O'_2$  respectively, and we have the relation  $O'_1 O'_2 = O_1 O_2 \sin \theta$ , or,

$$2b = 2d \sin \theta. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the angles  $\phi_1$  and  $\phi_2$  are as indicated, we also have

$$\sigma_1 \sin \phi_1 = \sigma_2 \sin \phi_2, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\sigma_1 \cos \phi_1 + \sigma_2 \cos \phi_2 = 2b, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which, combined with (2), gives

$$\sigma_1 \cos \phi_1 + \sigma_2 \cos \phi_2 = 2d \sin \theta. \quad . \quad . \quad . \quad . \quad (5)$$

The expression for  $A$ , the area of the projection, depends on whether the smaller sphere is completely eclipsed or not

by the larger sphere. For values of  $\theta$  from zero to  $\theta_0$ , given by the relation

$$\sin \theta_0 = \frac{\sigma_1 - \sigma_2}{2d}, \quad . . . . . (6)$$

$A$  is equal to  $\pi\sigma_1^2$  simply, because the small sphere is completely covered. Over the remaining range from  $\theta_0$  to  $\frac{\pi}{2}$  a variable crescent appears in addition. The area of this crescent is

$$\pi\sigma_2^2 - \sigma_1^2(\varphi_1 - \cos \varphi_1 \sin \varphi_1) - \sigma_2^2(\varphi_2 - \cos \varphi_2 \sin \varphi_2).$$

Now, since the axis  $O_1 O_2$  is oriented at random, the angle  $\theta$  occurs with a proportional frequency  $\sin \theta . d\theta$ , so that the mean area of projection for all possible orientations is given by

$$\begin{aligned} \bar{A} = \int_0^{\frac{\pi}{2}} \pi\sigma_1^2 \sin \theta . d\theta + \int_{\theta_0}^{\frac{\pi}{2}} \{ \pi\sigma_2^2 - \sigma_1^2(\varphi_1 - \cos \varphi_1 \sin \varphi_1) \\ - \sigma_2^2(\varphi_2 - \cos \varphi_2 \sin \varphi_2) \} \sin \theta . d\theta. \quad . . . (7) \end{aligned}$$

Partial integration reduces this to the form

$$\begin{aligned} \bar{A} = \pi\sigma_1^2 - \int_{\theta_0}^{\frac{\pi}{2}} \left( \sigma_1^2 \frac{d\varphi_1}{d\theta} + \sigma_2^2 \frac{d\varphi_2}{d\theta} \right) \cos \theta . d\theta + \int_{\theta_0}^{\frac{\pi}{2}} ( \sigma_1^2 \cos \varphi_1 \sin \varphi_1 \\ + \sigma_2^2 \cos \varphi_2 \sin \varphi_2 ) \sin \theta . d\theta, \quad . . . (8) \end{aligned}$$

which may be written, for the sake of brevity,

$$\bar{A} = \pi\sigma_1^2 - I + J. \quad . . . . . (9)$$

Elimination of  $\varphi_1$  and  $\varphi_2$  from  $I$  and  $J$  by means of equations (3) and (5) gives

$$J = \frac{1}{2} \int_{\theta_0}^{\frac{\pi}{2}} \{ 4\sigma_1^2\sigma_2^2 - (4d^2 \sin^2 \theta - \sigma_1^2 - \sigma_2^2)^2 \} \sin \theta . d\theta, \quad (10)$$

and

$$I = \int_{\theta_0}^{\frac{\pi}{2}} \frac{ \{ \sigma_1^2 - \sigma_2^2 \}^2 - 4d^2(\sigma_1^2 + \sigma_2^2) \sin^2 \theta \} \cos^2 \theta . d\theta }{ \{ 4\sigma_1^2\sigma_2^2 - (4d^2 \sin^2 \theta - \sigma_1^2 - \sigma_2^2)^2 \} \sin \theta }. \quad (11)$$

The quantities  $I$  and  $J$  can be expressed in terms of complete elliptic integrals by means of the substitution

$$\cos \theta = \cos \theta_0 \cos \psi. \quad . . . . . (12)$$

With this transformation we obtain

$$J = 2d \cos^2 \theta_0 \sqrt{\sigma_1 \sigma_2} \int_0^{\frac{\pi}{2}} \left( 1 - \frac{d^2 \cos^2 \theta_0}{\sigma_1 \sigma_2} \sin^2 \psi \right)^{\frac{1}{2}} \sin^2 \psi \cdot d\psi. \quad (13)$$

Or, writing  $k^2 = \frac{d^2 \cos^2 \theta_0}{\sigma_1 \sigma_2}$ , which is less than unity over the range  $\sigma_1 + \sigma_2 > 2d > \sigma_1 - \sigma_2$  contemplated,

$$J = \frac{2(\sigma_1 \sigma_2)^{\frac{3}{2}}}{d} \int_0^{\frac{\pi}{2}} k^2 \sin^2 \psi (1 - k^2 \sin^2 \psi)^{\frac{1}{2}} d\psi. \quad \dots \quad (14)$$

$$= \frac{2\sigma_1 \sigma_2}{3q} \{ (2k^2 - 1)E_1(k) + (1 - k^2)F_1(k) \},$$

where  $q$  is written for  $d \sqrt{\sigma_1 \sigma_2}$  and  $F_1(k)$  and  $E_1(k)$  are the complete elliptic integrals of argument  $k$ , and of the first and second orders respectively.

With the same substitution (12) the quantity  $I$  takes the form

$$I = \frac{(\sigma_1^2 - \sigma_2^2)^2 \cot^2 \theta_0}{4d \sqrt{\sigma_1 \sigma_2}} \int_0^{\frac{\pi}{2}} \frac{\cos^2 \psi \cdot d\psi}{(1 + \cot^2 \theta_0 \sin^2 \psi) \Delta(\psi)}$$

$$- \frac{(\sigma_1^2 + \sigma_2^2) \sqrt{\sigma_1 \sigma_2}}{d} \int_0^{\frac{\pi}{2}} \frac{k^2 \cos^2 \psi \cdot d\psi}{\Delta(\psi)}, \quad (15)$$

where  $\Delta(\psi) = (1 - k^2 \sin^2 \psi)^{\frac{1}{2}}$ ,  $k$  having the same value as previously. If we write  $n = \cot^2 \theta_0$  and integrate, we obtain

$$I = \frac{(\sigma_1^2 - \sigma_2^2)^2}{4q \sigma_1 \sigma_2} \{ (n+1) \Pi_1(n, k) - F_1(k) \}$$

$$- \frac{\sigma_1^2 + \sigma_2^2}{q} \{ E_1(k) - (1 - k^2) F_1(k) \}, \quad (16)$$

where  $\Pi_1(n, k)$  is the complete elliptic integral of the third order. The expression for  $\bar{A}$  in (9) therefore becomes, using (14) and (16),

$$\bar{A} = \pi \sigma_1^2 + \frac{\sigma_1^2 + \sigma_2^2}{q} \{ E_1(k) - (1 - k^2) F_1(k) \}$$

$$+ \frac{2\sigma_1 \sigma_2}{3q} \{ (2k^2 - 1) E_1(k) + (1 - k^2) F_1(k) \}$$

$$- \frac{(\sigma_1^2 - \sigma_2^2)^2}{4q \sigma_1 \sigma_2} \{ (n+1) \Pi_1(n, k) - F_1(k) \}. \quad (17)$$

Dividing both sides by  $\pi\sigma_1^2$ , and writing  $p=\sigma_2/\sigma_1$ , so that  $p < 1$ ,

$$\begin{aligned} \frac{\bar{A}}{\pi\sigma_1^2} = & 1 + \frac{p}{\pi q} \left[ E_1(k) \left\{ \frac{1}{p} + p + \frac{2}{3}(2k^2 - 1) \right\} \right. \\ & \left. - F_1(k) \left\{ (1 - k^2) \left( \frac{1}{p} + p - \frac{2}{3} \right) \right\} \right] \\ & - \frac{(1 - p^2)^2}{4\pi pq} \{ (n + 1)\pi_1(n, k) - F_1(k) \} \quad \dots (18) \end{aligned}$$

5. This expression enables us to evaluate the ratio of the mean collision area of the two spheres relative to that of the larger one only, for all values of  $\sigma_1, \sigma_2$  and  $d$  within the limits specified. In the special case when the two spheres are equal,  $p=1$ , so that the last term vanishes. In that case, also,  $q=d/\sqrt{\sigma_1\sigma_2}=d/\sigma$ , and  $\theta_0$  now being zero,  $k=d/\sigma$ . For two equal spheres there thus emerges the equation

$$\begin{aligned} \frac{\bar{A}}{\pi\sigma^2} = & 1 + \frac{1}{\pi k} \left\{ \frac{4}{3} E_1(k)(k^2 + 1) - \frac{4}{3} F_1(k)(1 - k^2) \right\} \\ = & 1 + \frac{4}{3\pi} \{ (1/k + k)E_1(k) - (1/k - k)F_1(k) \} \\ = & 1 + X, \quad \dots \dots \dots (19) \end{aligned}$$

which is the expression obtained previously and mentioned earlier.

It is worth noting that it provides the solution for any number of equal spheres which successively overlap one another, whether their centres are equally spaced or not. For clearly the excess over unity, *i.e.*,  $X$ , is the contribution to the mean area due to the crescent formed by the variable overlapping of the two spheres. If there are  $N$  equal spheres arranged on a straight line, there will be  $N-1$  such crescents, and, in order to find  $\bar{A}/\pi\sigma^2$ , we have merely to add extra terms of the same type as  $X$ . Thus, for  $N$  spheres

$$\bar{A}/\pi\sigma^2 = 1 + (X_{1,2} + X_{2,3} + X_{3,4} + \dots + X_{(N-1),N}). \quad (20)$$

In this expression  $X_{3,4}$ , for example, represents the contribution of the crescent formed by the third sphere overlapping the fourth, and in each such term the appropriate value of  $k$  must be inserted. Thus, in  $X_{3,4}$ , the value of  $k$  is  $\frac{d_{3,4}}{\sigma}$ , where  $2d_{3,4}$  equals the distance between the centres of the third and

fourth spheres. If all these distances are equal, *i.e.*, if the spheres are equally spaced,

$$\bar{A}/\pi\sigma^2 = 1 + (N-1)X \quad . \quad . \quad . \quad . \quad . \quad (21)$$

for all the crescents have the same area for the same orientation.

6. This specialised aspect of the subject may, with little difficulty, be made quite general, so as to include cases in which  $2d > 2\sigma$ . The difference introduced is that the integration has to be subdivided into two stages in which the projections do, and do not, overlap; otherwise the process is the same, and need not be repeated in detail. The result for two non-overlapping spheres of equal size proves to be

$$\begin{aligned} \frac{\bar{A}}{\pi\sigma^2} &= 1 + \frac{4}{3\pi} \left\{ \left( \frac{1}{c^2} + 1 \right) E_1(c) - \left( \frac{1}{c^2} - 1 \right) F_1(c) \right\} \\ &= 1 + Y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22) \end{aligned}$$

In this case  $c = \frac{\sigma}{d} < 1$ , and it will be noted that the expression differs from (19). The limiting value is 2, as we should expect, when  $c=0$ , *i.e.*, when the diameters of the spheres are negligible compared with their distance apart, with the result that the effect of overlapping is inappreciable and each sphere makes a contribution  $\pi\sigma^2$  independently to  $\bar{A}$ . When the spheres are just in contact, in which case  $c = \frac{\sigma}{d} = 1 = \frac{d}{\sigma} = k$ , (22) reduces to the same value as (19), *viz.*,  $1 + \frac{8}{3\pi}$ .

We may deduce the value of  $\frac{\bar{A}}{\pi\sigma^2}$  for any number of non-overlapping equal spheres in precisely the same way as indicated in the last paragraph, and obtain

$$\frac{\bar{A}}{\pi\sigma^2} = 1 + (Y_{1,2} + Y_{2,3} + Y_{3,4} + \dots + Y_{(N-1)N}) \quad . \quad (23)$$

with the appropriate values of  $c$  introduced. Or we may have a mixture of overlapping and non-overlapping spheres, and therefore use in the summation an  $X$  or a  $Y$  expression according as  $d/\sigma$  for the pair of spheres in question is less or greater than unity.

7. Returning now to the general case of two unequal spheres, as expressed in equation (18), we have to transform the last



term for purposes of calculation. Tables for the third order complete elliptic integrals,  $\Pi_1(n, k)$ , are not available, but these functions can be expressed, as is well known, in terms of complete and incomplete integrals of the first and second orders. The required relation, given by Legendre, is

$$\frac{(1-b^2 \sin^2 \theta_0)^{\frac{1}{2}}}{\sin \theta_0 \cos \theta_0} \Pi_1(n, k) = \frac{\pi}{2} + \frac{\sin \theta_0}{\cos \theta_0} (1-b^2 \sin^2 \theta_0)^{\frac{1}{2}} F_1(k) - E_1(k) F(b, \theta_0) + F_1(k) \{F(b, \theta_0) - E(b, \theta_0)\}, \quad (24)$$

where  $b^2 = 1 - k^2$ , and  $F(b, \theta_0)$  and  $E(b, \theta_0)$  are, respectively, first and second incomplete elliptic integrals of argument  $b$  and amplitude  $\theta_0$ . The insertion of this relation in the last term of (18), together with application of the connections between  $q$ ,  $p$ ,  $n$ , and  $\theta_0$ , gives the result

$$\begin{aligned} \frac{\bar{A}}{\pi \sigma_1^2} = & 1 + \frac{p}{\pi q} \left[ \left\{ \frac{1}{p} + p + \frac{2}{3}(2k^2 - 1) \right\} E_1(k) \right. \\ & \left. - (1 - k^2) \left( \frac{1}{p} + p - \frac{2}{3} \right) F_1(k) \right] \\ & - \frac{1 - p^2}{\pi} \left[ \frac{\pi}{2} + F_1(k) \{F(b, \theta_0) - E(b, \theta_0)\} - E_1(k) F(b, \theta_0) \right] \end{aligned} \quad (25)$$

Thus the expression is obtained in terms of functions for which tables are available. We may write (25) as

$$\frac{\bar{A}}{\pi \sigma_1^2} - 1 = Z \quad (26)$$

where  $Z$  represents the contribution to the mean area—in terms of  $\pi \sigma_1^2$ —of the variable crescent (Fig. 2) formed by the overlapping of sphere 1 over sphere 2. I have calculated  $Z$

for seven values of the ratio  $p = \frac{\sigma_2^2}{\sigma_1^2}$ , ranging from unity to 0.4.

This lower limit has been adopted because there is no evidence for the existence of relatively great differences of dimensions of molecules commonly in the gaseous state. The greatest and smallest atoms for which reliable estimates have been made are xenon and helium respectively. For these atoms the ratio  $\frac{\sigma_2^2}{\sigma_1^2}$  is  $\frac{0.96}{1.76} = 0.55$ . Possibly the hydrogen atom is somewhat smaller than that of helium, but not to such an extent that it would be outside the limits above mentioned. The

range of calculation is in each case for values of  $q$ , i.e.,  $\frac{d}{\sqrt{\sigma_1\sigma_2}}$ , lying within limits determined by  $\sigma_1 + \sigma_2 > 2d > \sigma_1 - \sigma_2$ , or extending from external to internal contact of the spheres. The results are shown graphically in Fig. 3. From these curves, if necessary by interpolation, the values of  $Z$  can be

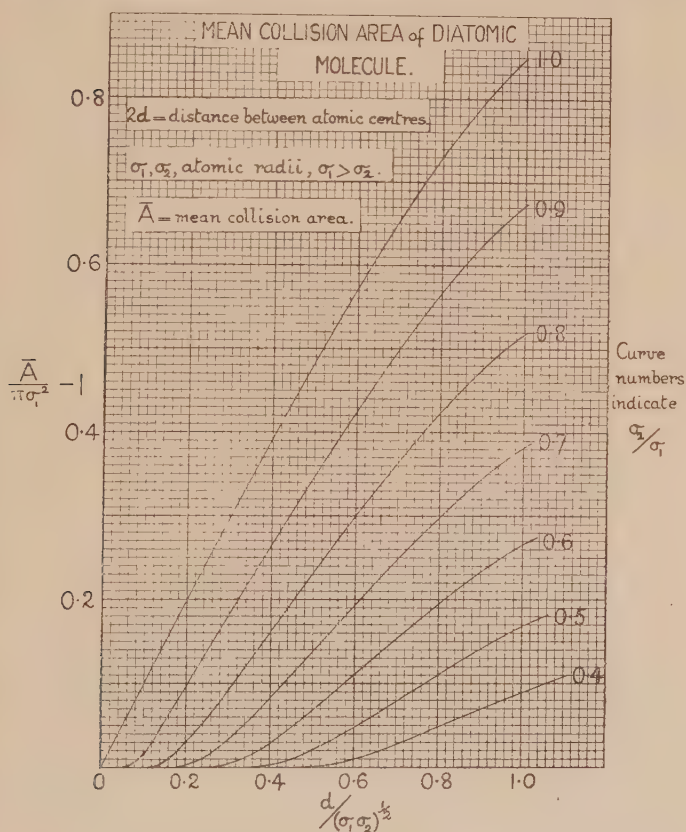


FIG. 3

obtained which correspond to specified values of  $d$ ,  $\sigma_1$  and  $\sigma_2$ . Or, conversely, we can derive  $q$  from given values of  $Z$ , and hence determine  $d$  if  $\sigma_1$  and  $\sigma_2$  are known.

8. With reference to these curves, there are several points of interest. For the particular case when  $p=1$ , equation (19) applies, and  $q=d/\sigma=k$ . The curve could be extended for

values of  $k$  greater than unity by means of formula (22), and  $\frac{\bar{A}}{\pi\sigma^2}-1$  would reach unit value asymptotically at  $k=\infty$ . For the other curves the corresponding extension has not been worked out—*i.e.*, we have no values for  $q > \frac{\sigma_1+\sigma_2}{2\sqrt{\sigma_1\sigma_2}}$ . But the

general courses which must be followed can be readily seen. For when  $q$  is very great, the effect of overlapping arises from a very small proportion of the possible orientations, and becomes negligible. Thus,  $\bar{A}$  approaches the limit  $\pi\sigma_1^2+\pi\sigma_2^2$ , or  $Z=p^2$ . Each curve, therefore, reaches asymptotically the value  $p^2$ ; for example, when  $p=0.5$ ,  $Z$  becomes 0.25 at  $q=\infty$ .

The use of this diagram is not limited to the model of two overlapping spheres. It also provides very simple means of solution in a large number of more complicated cases, such as, for example, that indicated in Fig. 4. Here there are two

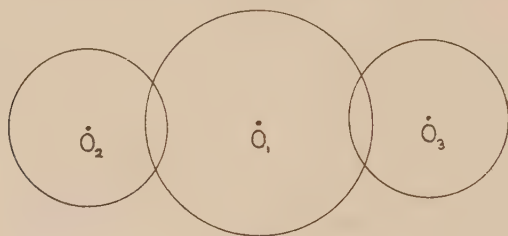


FIG. 4.

exactly similar bulges, one on each side of the large central sphere, which make equal and independent contributions to the mean area of projection. Thus  $\frac{\bar{A}}{\pi\sigma_1^2}-1$  becomes equal to  $2Z$ . More generally, if there are three spheres, all unequal, but rectilinearly disposed and successively overlapping, with the largest sphere occupying the middle position, we have

$$\frac{\bar{A}}{\pi\sigma_1^2}-1=Z_{1,2}+Z_{1,3}, \quad \dots \quad (27)$$

where  $Z_{1,2}$  and  $Z_{1,3}$  are the mean areas of the crescents formed by the overlapping of the largest sphere 1 over the outer smaller spheres 2 and 3 respectively. In  $Z_{1,2}$  and  $Z_{1,3}$ , of course, the proper values of  $q$  and  $p$  must be used.

There are other examples of extensions of this simple nature.

The simplicity appears to depend on two conditions—viz., (i.) the straight row of spheres must continually (but not necessarily uniformly) decrease in size as we pass from the largest one, either in a single direction or in two opposite directions, although, in the latter case, the decrements need not be equal on the two sides; (ii.) the spacings of the centres must be such that the successive total eclipses, which occur as the model is rotated towards the line of view, take place in the order of the sizes of the spheres, beginning with the smallest—*i.e.*, the outermost. With these conditions we can express the mean area of projection of the model, however many spheres it contains, by means of a series of independent  $Z$ 's, each of which is obtainable from the curves in Fig. 3.

9. A typical case in which these rules are departed from is shown in Fig. 5. Here we have a small sphere interposed

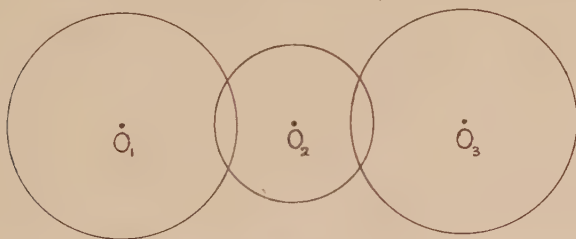


FIG. 5.

between two larger ones, and at a certain orientation its projection is completely covered by the two outer projections taken together, and not by each independently. The result is that Fig. 3 is no longer applicable. For it becomes necessary in evaluating  $\bar{A}$  to separate the integration into two stages which are not the same as those previously used, and the elliptic integrals obtained are not complete, as they are in (18). The formula is much more unwieldy, and its reduction to arithmetic much more laborious. It is not proposed, therefore, to deal here with cases of this type, except to say that the solution is derivable by the methods indicated, and that it can be applied numerically in special molecules suspected to have such shapes, and for which the necessary data become available.

10. It remains to consider for what actual molecules there is a reasonable prospect of obtaining the viscosity data for testing the applicability of the foregoing theory. Measure-

ments of viscosities at high temperatures are of great difficulty, and it is necessary to confine attention, for the moment at any rate, to chemical compounds which are ordinarily gaseous, or at least fairly volatile. It immediately appears that nearly all such molecules which satisfy the conditions—*i.e.*, have unequal atoms—have hydrogen as a constituent.

We have, in the first place, the group of hydrogen halides HF, HCl, HBr and HI, all of which are gaseous at ordinary temperatures. Following Lewis and Langmuir, these may be regarded as systems in which the single electron of the hydrogen atom has completed the corresponding inert gas arrangement of outer electrons around the halogen nucleus, the hydrogen nucleus occupying an external position. Thus, for example, HCl may be pictured as having the shape of the argon atom with the hydrogen nucleus as a protuberance. The work of W. L. Bragg and H. Bell\* has recently provided information regarding the distance between the nuclei of the hydrogen and the combined halogen, it being obtained simply by adding  $0.26 \times 10^{-8}$  cm. to the radius of the outer electron shell of the corresponding inert atom. But we do not yet know the correct way of regarding a single hydrogen atom from the point of view of molecular collision, since monatomic hydrogen is not available for viscosity measurements. If we may proceed upon the tentative assumption that it is spherical, a knowledge of the mean collision areas of the hydrogen halides deduced from viscosity data, in conjunction with the foregoing theory, will enable its magnitude to be estimated. That is to say, treating the halogen atom as a sphere having the collision radius ( $\sigma_1$ ) of the corresponding inert atom, and the hydrogen atom as an overlapping sphere of radius ( $\sigma_2$ ).  $\bar{A}$  determines the value of  $\sigma_2$  (Fig. 3). If this radius  $\sigma_2$  proves to be the same for all the hydrogen halides, it will indicate that the hydrogen protuberance behaves in molecular collisions as though it were spherical. The proportion of the mean collision area of a molecule attributable to a constituent hydrogen atom will thus be available for application to other cases.

There is also the group  $\text{H}_2\text{O}$ ,  $\text{H}_2\text{S}$  and, possibly,  $\text{H}_2\text{Se}$ , for which the necessary viscosity data should be obtainable. The water molecule is conceived by Lewis and Langmuir as a neon arrangement with two hydrogen nuclei, presumably on

\* W. L. Bragg and H. Bell. *Nature*, March 24, 1921.



opposite sides. In the  $\text{H}_2\text{S}$  molecule the arrangement would be the same, except that argon replaces neon, and in  $\text{H}_2\text{Se}$  the central configuration would be that of the krypton atom. All these are of the type indicated in Fig. 4, and, if we know the collision size of the hydrogen atom, Fig. 3 gives the theoretical mean collision area of the molecule in the manner already described. We should expect, for example, that the mean collision area of  $\text{H}_2\text{S}$  would be greater than that of  $\text{HCl}$  by the same amount as the mean collision area of  $\text{HCl}$  itself is greater than that of argon.

Yet another interesting group deserves consideration—namely,  $\text{NH}_3$ ,  $\text{PH}_3$ ,  $\text{AsH}_3$  and  $\text{SbH}_3$ . Presumably these have the external electron arrangement of Ne, Ar, Kr, and Xe respectively, with three hydrogen protuberances in each case. We may therefore regard each as a large central sphere with three smaller overlapping spheres attached, or rather, as behaving as such in molecular collision. This model is not strictly included in the above theoretical treatment, but it is clearly possible for the hydrogen spheres to be relatively so small and so disposed round the central one that they never can eclipse one another, except when the central sphere is also doing so. In that state of affairs—possibly reached in  $\text{SbH}_3$ , where the central sphere has the large size of xenon—the contributions to the mean collision area by the hydrogen protuberances would be independent of one another. We should thus expect that the mean collision area of  $\text{SbH}_3$  would be in excess of that of xenon by three times the excess of HI over xenon. For earlier members of the group, such as  $\text{NH}_3$ , this condition would probably not hold, and, owing to the effect of additional overlapping, the mean collision area should prove to be somewhat diminished. Until the effective collision radius of a hydrogen atom is known, one cannot estimate to what extent this would be the case.

There are, finally, the gaseous compounds of carbon and hydrogen. Nearly all of these probably have molecular shapes which are much more complicated than those contemplated in this Paper. In order to deal with them, further investigations of a theoretical nature will be necessary, and the analytical treatment is likely to prove very unwieldy, and perhaps even insoluble, owing to the removal of the simplification arising from the existence of an axis of symmetry. It seems probable that graphical methods will provide the most promising mode of attack.

Suitable molecules containing no hydrogen are very scarce. Sulphur dioxide and carbon bisulphide appear to be the only likely ones. The former is supposed to have the shape of a central argon atom with two neon atoms attached; the latter corresponds to a central neon atom and two outer argon atoms arranged as in Fig. 5. The validity of both these postulated models can be tested when the appropriate viscosity data become available.

11. In conclusion, I wish to express my thanks to Dr. H. Levy, of this College, for valuable hints regarding some of the mathematical aspects of this work.

#### DISCUSSION.

The author added that since the Paper was communicated, very satisfactory progress has been made in the Imperial College Physics Laboratory with the measurements of the viscosities of several gases to which the theory in the Paper would apply, including the group  $HCl$ ,  $H_2S$  and  $PH_3$ . The derived molecular dimensions pointed to the probability that the hydrogen atoms, instead being at equal distances from the central nucleus in all these compounds, became more and more remote as their number increased.

The President, Sir W. H. BRAGG, said he thought the idea that the hydrogen atoms were more widely separated in the more complex molecules was involved in a recent Paper by Sir J. J. Thomson. It may be possible to get at this by X-ray spectroscopic examination of some organic compounds. The molecules of some of these compounds must be very empty; in naphthalene for example, the hydrogen atoms probably act as bridges between the carbon atoms, so that it may be possible to get at the size of the hydrogen atom by X-ray methods.

XXXI. *An Electroculture Problem.* By C. CHREE, *Sc.D.*,  
*LL.D.*, *F.R.S.*

RECEIVED JUNE 2, 1921.

## ABSTRACT.

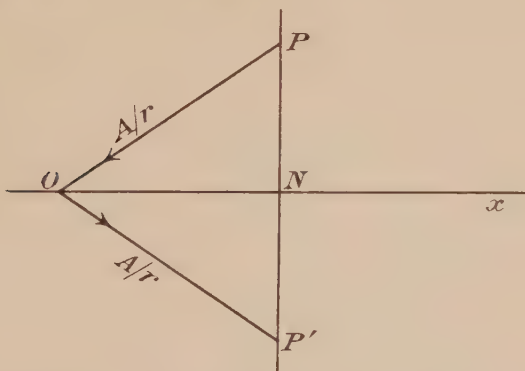
The problem is that presented by a number of fine parallel wires at the same height above the ground—treated as a plane surface—all charged to the same potential.

The chief object is to investigate how the potential gradient at ground level depends on the height and spacing of the wires. A second object is to find how the potential gradient varies with the height above the ground. The cases of a finite and an infinite number of wires are considered.

§1. THE problem considered is two-dimensional, being that presented by fine parallel wires at equal intervals  $2a$  apart, all at the same height  $h$  above the ground, and at the same electrical potential.

The electrostatic potential due to a single infinitely long wire in an infinite medium is  $A \log_e r$ , where  $r$  is the distance from the wire, and  $A$  is a constant, proportional to the electrical charge per unit length of the wire.

In practice the earth must be treated as a conductor at



zero potential. The figure is intended to represent a vertical plane perpendicular to the length of the wire,  $P$  being the section of the wire, and  $Ox$  in ground level. To maintain zero potential along  $Ox$ , we require a distribution of electricity on the ground, equivalent so far as points above ground level are concerned to a second charged wire parallel to the first, and at an equal distance from  $Ox$ . The charge per unit length of this image wire is numerically equal but of opposite sign to that in the real wire. In the figure  $P'$  represents the

section of the image wire,  $PP'$  being perpendicular to  $Ox$  and bisected at  $N$ . The electrical force at  $O$  is the resultant of  $A/r$  along  $PO$ , and  $A/r$  along  $OP'$ . The resultant is thus vertical and equals  $2Ah/r^2$ , or  $2Ah/(x^2+h^2)$ , where  $x=ON$ .

In an electroculture area there are a large number of parallel wires, each contributing to the vertical force at ground level. Let us suppose that the charge on any one of the more remote wires instead of being concentrated on the wire is uniformly distributed over a horizontal width  $a$  on either side of the wire. If this hypothesis is made for each of the more distant wires, we have a surface charge of uniform density at height  $h$ , extending from  $x_0-a$  to  $x'_0+a$ , where  $x_0$  and  $x'_0$  are the horizontal distances from  $O$  of the nearest and the furthest of the wires to which the hypothesis extends. The vertical force at  $O$  contributed by these more remote wires is then, according to the hypothesis, replaceable by the integral

$$\int_{x_0-a}^{x'_0+a} \frac{2Ah}{2a} \cdot \frac{dx}{h^2+x^2} \cdot \cdot \cdot \cdot \cdot \quad (1)$$

The value is thus  $(A/a) \left( \tan^{-1} \frac{x'_0+a}{h} - \tan^{-1} \frac{x_0-a}{h} \right)$ , or

$$(A/a) \left( \tan^{-1} \frac{h}{x_0-a} - \tan^{-1} \frac{h}{x'_0+a} \right) \quad (2)$$

The expression inside the bracket obviously represents the angle subtended at  $O$  by the hypothetical distribution.

§2. Comprehension may be aided by discussing a particular case. Suppose, then, we have in all 21 wires, and wish to find the force under the central wire at ground level. The origin being supposed under the central wire, the horizontal distances of the 10 wires on one side are  $2a, 4a \dots 20a$ . Each of the 21 wires has an image. The resultant vertical force at the origin due to the whole 42 wires, real and imaginary, is easily seen to be

$$\frac{2A}{h} \left[ 1 + 2 \left\{ \frac{1}{1+(2a/h)^2} + \frac{1}{1+(4a/h)^2} \cdot \cdot \cdot + \frac{1}{1+(20a/h)^2} \right\} \right].$$

It is easy evaluating this for any convenient value of  $h/a$ . If, for instance,  $h=2a$ , we have only to sum

$$\frac{1}{a} \left[ 1 + 2 \left( \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \right) \right].$$

To four decimal places, this is  $(A/a) \times 2.9636$ .

The contribution from the 10 most remote wires—five on either side—viz.,

$$\frac{2A}{a} \left( \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \right) \text{ equals } \frac{A}{a} \times 0.1690.$$

Suppose, however, that instead of regarding the charge as concentrated on these outside wires, we distribute it uniformly. It will then cover a patch on either side extending from  $x=11a$  to  $x=21a$ . Taking the contributions from both patches, from the integrals we get  $2(A/a)[\tan^{-1}(21a/h) - \tan^{-1}(11a/h)]$  or  $2(A/a)[\tan^{-1}10.5 - \tan^{-1}5.5]$ . The angles, of course, are supposed to be in circular measure. But we have only to multiply by 0.017453 the angles as expressed in degrees in an ordinary book of logarithms, where we get

$$\tan^{-1}10.5 = 84^{\circ}33'.58 \equiv 84^{\circ}.560,$$

$$\tan^{-1}5.5 = 79^{\circ}41'.71 \equiv 79^{\circ}.695.$$

Thus the contribution from the integral is

$$2(A/a)4.865 \times 0.017453, \text{ or } (A/a) \times 0.1698.$$

It is, of course, more accurate and quite as easy in this case to obtain the contribution from the 10 outside wires direct from the series as has been done above. But our immediate object is to illustrate the integration method, and show its accuracy.

The ultimate object of the integration method was really to bring the case of an infinite number of wires readily within our reach, with an accuracy sufficient for practical purposes. Suppose, then, we have obtained a result for 21 wires, and wish to find the addition to be made to this when the number becomes infinite. Taking the integral with the superior and inferior values of  $x$  as  $\infty$  and  $21a$  respectively, we have for the sum of the contributions from both sides

$$2(A/a)\{\tan^{-1}\infty - \tan^{-1}11.5\} = (A/a) \times 0.1899.$$

The use of the integral increases in accuracy as the inferior limit of  $x$  increases. When the inferior limit was  $11a$  the error, as we have seen, was less than  $\frac{1}{2}$  per cent. With an inferior limit of  $21a$  the accuracy will be much better, and we may feel assured that while the result  $(A/a) \times 0.1899$  obtained above may not be accurate in the fourth decimal place, it is exact enough for all ordinary practical purposes. Adding it to the result  $(A/a)(2.9636)$  obtained above for the 21 wires, we obtain for the force at ground level directly under a wire



of an infinitely extended electroculture area, having  $h=2a$ , the value  $(A/a)(3.1535)$ . It is hoped that this will sufficiently explain the analytical methods employed in calculating the force at ground level. The only difference in the procedure when the force is calculated for a point midway between two wires is that the estimate is made in the first instance for the nearest 20 wires. The contribution from the more remote wires on either side when their number is infinite is derived from the result of integration, viz.  $(2A/a)(\tan^{-1} \infty - \tan^{-1} 20a/h)$ .

§3. The vertical force could easily have been calculated at a number of different heights above ground level, but it appeared sufficient to calculate it at the half-height of the wires, and to obtain in addition its mean value between this height and ground level. The vertical force at height  $h/2$  due to one wire at horizontal distance  $x$  and its image wire is

$$A \left\{ \frac{h/2}{x^2 + (h/2)^2} + \frac{3h/2}{x^2 + (3h/2)^2} \right\} \dots \dots \dots (3)$$

Putting  $x=0$ ,  $2a \dots$  or else  $x=a$ ,  $3a \dots$  we get results applicable to a point directly under a wire or midway between two wires.

When the point is immediately under a wire the correction to allow for the difference between 21 wires and an infinite number is—

$$(A/a)(0.017453) \{180 - \tan^{-1}(42a/h) - \tan^{-1}(14a/h)\},$$

the angles being expressed in degrees.

For a point midway between two wires the allowance for the difference between 20 wires and an infinite number is similarly—

$$(A/a)(0.017453) \{180 - \tan^{-1}(40a/h) - \tan^{-1}(13.3a/h)\}.$$

§4. If  $V_0$  denote the potential at height  $z$ , the vertical force, numerically considered, is  $\frac{dV}{dz}$ . Thus the mean value of the force between ground level and the height  $h/2$  is—

$$V_{h/2} \div (h/2),$$

where  $V_{h/2}$  represents the potential—which varies of course with  $x$ —at the height  $h/2$ .

The potential at a height  $h/2$  in the vertical through the origin, due to a single wire at a horizontal distance  $x$  and its image combined, is—

$$(A/2) \log_e \frac{x^2 + 9h^2/4}{x^2 + h^2/4} \dots \dots \dots (4)$$

Thus, for the potential at height  $h/2$  directly under the central wire of a series of 21 we have, when  $h=2a$ —

$$V_{h/2} = \frac{1}{2} A \log_e 10 \left[ \log 9 + 2 \left\{ \log \frac{13}{5} \times \frac{25}{17} \times \frac{45}{37} \times \frac{73}{65} \times \frac{109}{101} \times \frac{153}{145} \right. \right. \\ \left. \left. \times \frac{205}{197} \times \frac{265}{257} \times \frac{333}{325} \times \frac{409}{401} \right\} \right]. \quad (5)$$

All the logarithms inside the square bracket are to base 10. The separate factors  $13/5$ ,  $25/17$ , &c., represent the contributions from the successive 10 wires (and their images) on each side of the central wire.

An allowance is easily made for the more remote wires. The contribution to  $V_{h/2}$  from a single wire and its image as given above in (4) may be written

$$(A/2) \{ \log_e(1 + 9h^2/4x^2) - \log_e(1 + h^2/4x^2) \} \quad . \quad (6)$$

Expanding in powers of  $h/x$ , and retaining the two first terms, we get—

$$A \left( \frac{h^2}{x^2} - \frac{5}{4} \frac{h^4}{x^4} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now, suppose, as before, the charge spread over a width  $2a$  for each of the more remote wires, then for the contribution to the potential from the remote wires on one side of the origin we get from (7)—

$$\int_{x_0-a}^{x_0'+a} (A/2a) \left( \frac{h^2}{x^2} - \frac{5}{4} \frac{h^4}{x^4} \right) dx, \quad . \quad . \quad . \quad . \quad (8)$$

where  $x_0$  and  $x_0'$  have the same significance as in (2).

Doubling the result of the integration, viz.—

$$\left[ (A/2a) \left( \frac{-h^2}{x} + \frac{5}{12} \frac{h^4}{x^3} \right) \right]$$

taken between the limits, we get the contribution to  $V_{h/2}$  from the more remote wires on both sides of the origin, and dividing the value of  $V_{h/2}$  so found by  $h/2$ , we get the mean value of the vertical force between ground level and the height  $h/2$ .

As an example, suppose we have found the mean force due to the nearest 20 wires up to height  $h/2$ , midway between two wires, and wish to find the addition to be made when the number of wires becomes infinite. The limiting values of  $x$  in the integral are  $20a$  and  $\infty$ , so its value, when doubled and divided by  $h/2$  is simply

$$\frac{2A}{a} \left\{ \frac{h}{20a} - \frac{5}{12} \left( \frac{h}{20a} \right)^3 \right\} \equiv \frac{Ah}{10a^2} \left\{ 1 - \frac{1}{960} \left( \frac{h}{a} \right)^2 \right\} \quad . \quad (9)$$

Unless  $h/a$  is large, the first term gives quite a good approximation. When  $h=2a$  we have  $(A/a)0.2(1-1/240)=0.1992A/a$ .

§ 5. The original investigation had two objects in view :—

1. To ascertain whether an experimental station with only a few wires could supply information likely to be of practical service.

2. To ascertain the spacing of wires appropriate to electro-culture areas.

Obviously if the wires are far apart, and near the ground, the potential gradient at ground level will be much higher at points under a wire than at points midway between wires. If a very high potential gradient were injurious, and a moderate gradient beneficial, we might have improvement of one part of the crop neutralised by injury to another. It should at least facilitate drawing conclusions, as to the merits of electro-culture, if an approach to uniformity of conditions is secured throughout the area.

In the original calculations a number of values of  $h/a$  were considered, but here results will be given mainly for three values only, viz., 1, 2 and 4, as these sufficiently illustrate the main features.

Table I. contrasts the exact results obtained from the series or the logarithms with the approximate values obtained by integration for the contribution to the force arising from the most remote 10 of a series of 21 or of 20 wires. It is confined to the values of the force at ground level, and to the mean values of the force between ground level and the half-height of the wires.

TABLE I.—*Force due to the 10 most remote wires (Coefficient of  $A/a$ ).*

Force under central wire when 21 wires.					Force midway between central wires when 20 wires.			
At "ground level."			Mean up to $h/2$		At "ground level."		Mean up to $h/2$ .	
$h/a$	True.	By Integral.	True.	By Integral.	True.	By Integral.	True.	By Integral.
1	0.0857	0.0861	0.0857	0.0860	0.0989	0.0391	0.0988	0.0993
2	0.1690	0.1698	0.1682	0.1689	0.1944	0.1954	0.1933	0.1942
4	0.3198	0.3211	0.3148	0.3120	0.3646	0.3662	0.3581	0.3533

The difference between the exact and approximate values

for a given value of  $a$  increases, as was to be expected, with the value of  $h$ . But even for  $h/a=4$  this difference is a very insignificant fraction of the force due to the nearest 21 or 20 wires. For higher values of  $h/a$  it would be an improvement to extend the exact calculation to a larger number of wires than 21 or 20, especially when considering the force or potential at heights above ground level.

The principal object of calculating Table I. was to show the degree of accuracy attainable by the approximate methods of integration.

Table II. gives the results obtained, when the number of wires is infinite, for the force arising from all more remote than the nearest 21 or 20. These results being obtained by the integration methods, are all approximate. The accuracy to be expected is considerably higher, however, than for the corresponding integration results in Table I. For a given value of  $a$  the accuracy will naturally diminish as  $h$  increases. Also the numerical value of the contribution from the more remote wires varies approximately as  $h$  when  $a$  is constant. It is roughly independent of the height above ground level, so long as the half-height of the wire is not exceeded.

TABLE II.—*Contribution to force from wires outside the central 21 or 20, when number is infinite (Coefficient of  $A/a$ ).*

$h/a$ .	Under a wire. Additional to effect of 21			Midway between wires. Additional to effect of 20		
	1	2	4	1	2	4
At "ground level" ...	0.0952	0.1899	0.3764	0.0999	0.1994	0.3948
Mean between "ground level" and $h/2$ ...	0.0951	0.1898	0.3752	0.0999	0.1992	0.3933
At $h/2$ ... ..	0.0951	0.1895	0.3732	0.0998	0.1988	0.3911

It will be observed that each numerical result in Table II. is in excess of the corresponding result in Table I., but the excess is not large.

§ 6. Table III. summarises the results obtained for the vertical force at points exactly under a wire, or exactly midway between two wires. It is convenient to express the force in terms of  $A/a$ , but it should be remembered that for a given value of  $h/a$  an increase in  $h$  means a corresponding increase in  $a$ . Increasing the height, keeping the span-height ratio constant, always entails a fall of force.

It has also to be remembered that by "ground level" in the table is really meant the surface of zero potential. It is thus to be interpreted as the top of the crop, not the actual surface of the ground in an electroculture area. If, for example, the wires over a field of wheat are 8 ft. apart and 8 ft. above the ground,  $a$  is 4 feet throughout the season; but  $h$  diminishes from 8 ft. at the start to 4 ft. at the finish, if the wheat grows to 4 ft. high. Under these circumstances, we should have  $h/a$  varying from 2 to 1, and if we had only a single wire, and maintained its potential constant, the vertical force at crop level would double in the course of the season.

TABLE III.—*Vertical force due to overhead wires (Coefficient of  $A/a$ ).*

		Directly under a wire (maximum).				Midway between wires (minimum).				
Number of wires ...		1	5	21	$\infty$	2	4	20	$\infty$	
Value of $h/a$	Level of point.									
1	At "ground level"	2.000	3.035	3.330	3.425	2.000	2.400	2.781	2.881	3.000
	Mean up to $h/2$ ...	2.197	3.201	3.495	3.590	1.911	2.303	2.682	2.782	3.000
	At $h/2$ ...	2.667	3.604	3.896	3.991	1.723	2.098	2.473	2.573	3.000
2	At "ground level"	1.000	2.400	2.964	3.154	1.600	2.215	2.931	3.130	3.000
	Mean up to $h/2$ ...	1.099	2.440	2.996	3.186	1.609	2.197	2.900	3.100	3.000
	At $h/2$ ...	1.333	2.553	3.093	3.282	1.600	2.133	2.813	3.011	3.000
4	At "ground level"	0.500	1.800	2.765	3.142	0.941	1.581	2.747	3.142	3.000
	Mean up to $h/2$ ...	0.549	1.832	2.767	3.143	1.001	1.622	2.747	3.141	3.000
	At $h/2$ ...	0.667	1.897	2.773	3.147	1.124	1.699	2.745	3.136	3.000

If the effect, whether beneficial or otherwise, depends only on the potential gradient at crop level, the "ground level" values in Table III. alone matter. If  $h/a$  is 2 or larger—*i.e.*, if the distance apart of the wires is not greater than their height—the difference, when the number of wires is infinite, between the force directly under a wire (*i.e.*, the maximum) and the force midway between two wires (*i.e.*, the minimum) is less than 1 per cent. of their mean. For practical purposes, the force may then be regarded as everywhere the same and equal to  $\pi A/a$ .

Even with  $h/a$  as small as 1—*i.e.*, with the height only half the distance apart of the wires—when the number of wires is infinite, the least value of the force at "ground level" is 0.81 of the largest value. With further reduction in  $h/a$  the ratio of the minimum to the maximum value falls rapidly.



For values  $\frac{3}{4}$ ,  $\frac{1}{2}$  and  $\frac{1}{4}$  of  $h/a$ , the respective values of the ratio are only 0.68, 0.43 and 0.18 approximately.

§ 7. An infinite number of wires represents, of course, an ideal state, from which a considerable fraction of any real electroculture area must differ sensibly. As an example, suppose we travel outwards in the direction perpendicular to the wires from the centre of an electroculture area in which  $h=2a$ . Even at first, the value of the force at ground level is not everywhere absolutely the same, but it varies only from  $3.15 A/a$  under a wire to  $3.13 A/a$  midway between adjacent wires. As we approach the margin, the difference between the force under a wire and the force midway between adjacent wires increases, and, at the same time, both values fall off. When we are directly under the last wire the force has fallen to about  $2.1 A/a$ . When we pass outside the area, travelling in the same direction, the force continually diminishes, but it remains finite for quite a considerable distance. It is still about  $0.1A/a$  when we have got to a distance of  $20a$ , or  $10h$ , from the margin.

At a considerable distance outside the area, the force for a given spacing and voltage of the wires varies approximately as the apparent height of the nearest wire. Thus, the distance to which an appreciable potential gradient effect extends increases as the height of the overhead system.

Here we have strictly confined ourselves to the electrostatic problem presented when all the charge is on the wires, and there is no appreciable excess of ions of one sign in the atmosphere. When there is appreciable free charge, with an excess of ions of one sign, the potential gradient is affected. When the overhead voltage is high enough, it must inevitably produce copious ionisation near the wires, and a strong wind will spread the ionisation to quite a long distance to leeward of the area. These and other practical considerations must be borne in mind by anyone wishing to apply the present calculations. Perhaps the most immediately useful deduction from them is that, apart from wind effects, a very uniform set of conditions can be secured at crop level if the distance between adjacent wires does not exceed the height of the wires above the crop. Also, under these conditions, the potential gradient is nearly uniform from the crop level up to at least the half height of the wires above the crop. This last point was really investigated with an eye to certain experimental investigations.

## DISCUSSION.

Mr. E. V. APPLETON (communicated): I presume that Dr. Chree is aware that Maxwell ("Electricity and Magnetism," Vol. I.) has dealt with the electrical theory of a charged grating of parallel wires. In Maxwell's treatment, however, the assumption is made that the distance between the wires and the conducting plane (the ground in this case) is large compared with the distance between successive wires, and in such a case it is a simple matter to deduce from Maxwell's equations that the potential gradient at the ground level is equal to  $\frac{V}{h + \frac{a}{\pi} \log_e \frac{a}{\pi c}}$ , where  $V$  is the maintained potential difference

between the wires and the ground and  $c$  is the radius of the cylindrical wires. But Dr. Chree's treatment of the problem gives us information about the cases in which the distance between the wires and the ground is comparable with the distance between successive wires.

Workers on the physics of the triode vacuum tube meet the same type of problem in dealing with the action of a charged grating in producing a potential gradient near the cathode of such a tube. It has usually been assumed that Maxwell's equations give a fair approximation to the truth in the practical case when the distance between the cathode and grating is about equal to the distance between the grating wires. Dr. Chree shows us how far such an assumption is justifiable.

Mr. F. J. W. WHIPPLE (partially communicated): I find that I was mistaken in thinking that the problem of the grating of parallel wires is one of those discussed in Sir J. J. Thomson's "Recent Researches in Electricity and Magnetism." The problem is dealt with by Maxwell himself ("Electricity and Magnetism," Vol. I., § 233). Maxwell's discussion is only directly applicable to the case in which the distance between consecutive wires is small compared with their distance from the conducting plane. The method of conjugate functions can be used effectively in working out the more general problem and leads to the following results:

If  $\frac{1}{2}A$  be the line density of the charge on each of a very large number of thin parallel wires at height  $h$  above the ground, and if  $2a$  be the distance between adjacent wires, then the vertical force,  $F$ , at a point on the ground at a horizontal distance,  $x$ , from one of the wires is given by the formula

$$F = \frac{\pi A}{a} \cdot \frac{\sinh \frac{\pi h}{a}}{\cosh \frac{\pi h}{a} - \cos \frac{\pi x}{a}}.$$

The force varies between the values

$$\frac{\pi A}{a} \coth \frac{\pi h}{2a} \quad \text{and} \quad \frac{\pi A}{a} \tanh \frac{\pi h}{2a}.$$

It will be seen that for the variation from the mean not to exceed 10 per cent. on either side  $\coth \frac{\pi h}{2a}$  must not be greater than 1.1, and therefore  $h$  must not be less than  $0.97a$ . The condition is satisfied if the height of the wires is at least half the distance between them.

It is easy to verify that the values given by Dr. Chree in Table III. for the vertical force at ground level with an infinite number of wires are in accordance with my formula.

The formula for  $V$ , the potential difference between the wires and the ground, involves  $r$ , the radius of the cross section of a wire; it is

$$V = A \cdot \log \left[ \frac{2a}{\pi r} \sinh \frac{\pi h}{a} \right].$$

For comparison with the case of a single charged wire this formula may be written

$$V = A \log \frac{2h}{r} + A \log \frac{\sin \pi h/a}{\pi h/a},$$

and for comparison with the case when a uniformly charged conductor is substituted for the wires it takes the form

$$V = \frac{\pi A h}{a} + A \log \frac{a}{\pi r} + A \log \left( 1 - e^{-2\pi h/a} \right).$$

Dr. CHREE communicated the following reply: The experimental electro-culture area for which I first made calculations had only five wires. In any actual case only an approach is made to an infinite number. Thus a method was essential which applied to a finite number of wires. In the course of the investigations I hit on a simple method, with an obvious geometrical interpretation, which enabled me to take account of remote wires, and so to include the infinite number as a limit. I thought an exact mathematical solution for an infinite number probably existed somewhere, but having satisfied all my immediate wants I did not pursue the enquiry. Thirty years ago I read the proofs of the last edition of "Maxwell's Electricity" (and also of Sir J. J. Thomson's "Recent Researches"), but I had forgotten the existence of the solution\* to which Mr. Appleton refers. The solution for an infinite number of wires should be specially useful for obtaining the exact variation of force at ground level with distance from the nearest wire in cases where  $a/h$  considerably exceeds 1. In applications of the mathematical formulæ, especially those involving the radius of the wire, and particularly to points near the wire, it should be remembered that with the usual fine wires there is a considerable sag, and there may be a corona.

\* 203 *et seq.*, and Fig. XIII.

*Demonstration of Some Optical and Acoustical Experiments.*  
By PROF. C. V. RAMAN, *University of Calcutta.*

SHOWN AT THE MEETING OF JUNE 24TH, 1921.

1. The ballistic sonometer, an instrument which indicates the intensity of a noise by the extent of the movement of a spot of light.

2. Vibrations of stringed instruments. By means of a small optical lever in contact with the bridge of a 'cello and a revolving mirror the vibrations of the bridge under various conditions were shown on the screen. The effects of meeting and of playing the "wolf" note were demonstrated.

3. The modes of vibration of a Trevelyan rocker were exhibited in the same way.

4. Indian drums, 10 centuries old, in which the membrane is loaded in such a way as to produce harmonic overtones were exhibited, and their modes of vibration when struck in different ways exhibited by means of sand figures.

5. The polygonal form of an oscillating liquid drop was projected on the screen.

6. A robust ripple apparatus was described, and photographs shown illustrating the change of phase of waves on passing through a focus.

7. Various interference and diffraction effects shown when light is transmitted by heterogeneous media were exhibited.

8. A spectrum traversed by Talbot's bands was shown, to illustrate the sudden change of colour between the bands.

*The Stability of Atoms.* By SIR ERNEST RUTHERFORD,  
F.R.S. (*Abridged report of a Lecture delivered before the  
Society on June 10, 1921.*)

DURING the latter half of the nineteenth century it was generally accepted that the atoms of the chemist and physicist were permanent and indestructible, and were uninfluenced by the most drastic physical and chemical agencies available. The existence of elements in the earth that appeared to have suffered no certain change within periods of time measured by the geological epochs gave a strong support to the prevailing view of the inherent stability of the elements. The discovery at the beginning of the twentieth century that the radio-active elements uranium and thorium were undergoing a veritable transformation, spontaneous and quite uncontrollable by the agencies at our disposal, was the first serious shock to our belief in the permanency of the elements. The essential phenomena which accompanied the series of transformations soon became clear. The disintegration of an atom was accompanied either by the emission of a swift atom of helium carrying a positive charge, or of a swift electron. With the exception possibly of potassium and rubidium, only the heavy radio-active elements showed this lack of stability. The great majority of the chemical elements appeared, as before, to be inherently stable and to be unaffected by the most intense forces at our disposal.

A number of attempts have been made from time to time to test whether the atoms of the elements can be broken up by artificial methods. Some thought that they had obtained evidence of the production of hydrogen and of helium in the electric discharge tube. It is, however, a matter of such great difficulty to prove the absence of these elements as a contamination in the materials used that the evidence of transformation has not carried conviction to the minds of the majority of scientific men.

In this lecture an account will be given of some preliminary experiments which indicate the possibility of artificial disintegration of some of the ordinary elements by a new method. Before discussing the results, it is desirable to say a few words on the modern conception of the structure of the atom. The results have been interpreted on the nuclear theory of atomic constitution. According to this view, the atom is to be



regarded as consisting of a minute positively charged nucleus, in which most of the mass of the atom is concentrated, surrounded at a distance by a distribution of negative electrons which make the atom electrically neutral. We know that one or more of these outer electrons can be easily removed from the atom. The atom thus undergoes a kind of transformation, but only a temporary one, for the missing electrons are readily recaptured from neighbouring atoms. It seems not unlikely that the whole of the exterior electrons might be removed from an atom without interfering sensibly with the stability of its nucleus. Under suitable conditions, the atom would promptly regain its lost electrons and be indistinguishable from the original atom. In order to produce a *permanent* transformation of the atom, it is necessary to disintegrate the nucleus. Such a disruption of the nucleus occurs spontaneously in the radio-active atoms, and the processes appear to be irreversible under ordinary conditions.

The nucleus, however, is very small, and its constituent parts are probably held together by strong forces; and only a few agencies are available for an attack on its structure. The most concentrated source of energy at our command is a swift  $\alpha$ -particle, and it is to be expected that an  $\alpha$ -particle in its flight would occasionally approach so close to the nucleus as to disintegrate its structure. It is, indeed, from a study of the deflexion of swift  $\alpha$ -particles in passing through matter that we have obtained the strongest evidence in support of the theory of the nuclear constitution of atoms. In the region surrounding a heavy nucleus, the inverse square law holds for the forces of repulsion between the charged  $\alpha$ -particle and positively charged nucleus. The particle describes a hyperbolic orbit round the nucleus, and the amount of its deflexion depends on the closeness of its approach. It is from a study of this scattering of  $\alpha$ -particles, combined with Moseley's interpretation of the X-ray spectra of the elements, that we know the magnitude of the resultant positive charge on the nucleus. This charge, in fundamental units, is equal to the atomic or ordinal number of the element, and varies between 1 for hydrogen and 92 for uranium. Recently Chadwick has shown by direct measurements of the scattering of  $\alpha$ -particles that the charge on the nucleus is in close accord with Moseley's deduction, and has thus verified the correctness of this fundamental conclusion.

Some information about the dimensions of the nucleus can

be obtained by studying the amount of scattering of  $\alpha$ -particles at large angles by different atoms. The general results indicate that the nucleus of a heavy atom, if assumed spherical, cannot have a radius greater than  $6 \times 10^{-12}$  cm. It is not unlikely that the dimensions may be smaller than this. No doubt the size of a nucleus decreases with its atomic mass, and it is to be expected that the nucleus of the light elements should be smaller than for the heavy atoms. It is thus clear that the volume occupied by the nucleus is exceedingly small compared with that occupied by the atom as a whole.

A direct collision of an  $\alpha$ -particle with this minute nucleus is thus a rare occurrence. It can be estimated that even in the case of heavy elements only one  $\alpha$ -particle in about 10,000 makes a close collision with the nucleus. On account of the powerful repulsive field of the latter, the  $\alpha$ -particle may either be turned back before reaching the nucleus, or be so diminished in energy that it is unable to effect its disruption. The case of the lighter elements, however, is much more favourable; for the repulsive forces are so much weaker that we may expect the  $\alpha$ -particle to enter the nucleus structure without much loss of energy, and thus to be an effective agent in promoting the disintegration of the atom.

One of the most interesting cases to consider is that in which an  $\alpha$ -particle (helium nucleus) collides with the nucleus of an hydrogen atom. Marsden showed that in some cases the  $H$ -atom is set in such swift motion that it can be detected by the scintillation produced on a zinc sulphide screen. The maximum speed obtainable is 1.6 times that of the incident  $\alpha$ -particle, and such a swift  $H$ -atom is able to travel four times as far as the  $\alpha$ -particle before being stopped. For example, the maximum range of a  $H$ -atom set in motion by an  $\alpha$ -particle from radium  $C$ —range 7 cm. in air—corresponds to 29 cm. of air.

A close examination of the production of swift  $H$ -atoms by this method showed that the number was about 30 times greater than that to be expected if the colliding nuclei behaved as point charges repelling each other according to the inverse square law. This, and other observations, show that the law of the inverse square ceases to hold in such intense collisions, where the closest distance of approach is of the order  $3 \times 10^{-13}$  cm. It is probable that this distance is comparable with the actual dimensions of the structure of the  $\alpha$ -particle itself. Some recent experiments by Chadwick and Bieler indicate that

there is an abrupt change in the law of force at distances of about  $5 \times 10^{-13}$  cm. So far, no definite evidence has been obtained as to the nature of these forces which arise in the close collisions between nuclei. Attention should be directed to the enormous intensity of the electrical forces that come into play in such close collisions—forces much greater than can be brought to bear on an atom by ordinary laboratory methods. Unless the nucleus is a very stable structure, it is to be anticipated that it should be greatly disturbed, if not disintegrated, under the influence of such intense forces.

We must now consider the experiments which indicate that some of the lighter elements can be disintegrated by the action of  $\alpha$ -particles. When a stream of  $\alpha$ -particles is passed through dry air or nitrogen, a number of scintillations are observed far beyond the range of the  $\alpha$ -particle. These scintillations are due to swift charged particles which are bent in a magnetic field like  $H$ -atoms set in swift motion by  $\alpha$ -particles, and which, indeed, are undoubtedly  $H$ -atoms. Since this effect is not observed in dry oxygen or carbon dioxide, it appears likely that some of the nuclei of nitrogen have been disintegrated by the action of the  $\alpha$ -particles. Recently these experiments have been repeated by Mr. Chadwick and myself under much better optical conditions for counting these comparatively weak scintillations. It has been found that, using radium  $C$  as a source of  $\alpha$ -rays, the maximum range of the  $H$ -atoms from nitrogen atoms corresponds to 40 cm. of air, while the maximum range of the  $H$ -atoms from hydrogen, or any combination of hydrogen, is only 29 cm. under similar conditions. This result negatives the possibility that the presence of these  $H$ -atoms can be ascribed to any hydrogen contamination in the ordinary chemical sense.

This observation opened up a simple method of examining other elements besides nitrogen. Experiments were in all cases made beyond the maximum range (29 cm.) of ordinary  $H$ -atoms, so as to be quite independent of the presence of free or combined hydrogen in the material under examination. In this way it has been found that similar particles are produced in boron, fluorine, sodium, aluminium, and phosphorus. No definite effect has so far been observed for other elements of the production of particles with ranges greater than 29 cm. of air. The question of the production of slower velocity  $H$ -atoms has not so far been examined. The range of penetration of the atoms from aluminium is

specially marked, being more than 80 cm. While no definite proof has yet been obtained of the nature of these ejected particles, it seems probable that they are in all cases *H*-atoms liberated from the nuclei of the elements in question. It is of special interest to note that *H*-atoms are only liberated in elements whose mass is given by  $4n+2$  or  $4n+3$  where  $n$  is a whole number. No *H*-atoms are observed from elements like carbon, oxygen, and sulphur, whose mass is given by  $4n$ . This is an indication that the  $\alpha$ -particles are unable to liberate *H*-atoms from elements composed entirely of helium nuclei, but are able to do so from some elements composed of *H*-atoms as units as well as helium nuclei. It would appear as if the *H*-atoms were satellites of the main nuclear system and that one of them gained sufficient energy from a collision with an  $\alpha$ -particle to escape from its orbit with a high speed. If the long-range particles from aluminium are *H*-atoms, it can be calculated that the maximum energy of motion is somewhat greater than that of the incident  $\alpha$ -particle, indicating that the escaping fragment of the atom has gained energy from the system. It is of special interest to note that, in the case of aluminium, the direction of escape of the *H*-atom is to some extent independent of the direction of the  $\alpha$ -particle. Nearly as many are shot in the backward as in the forward direction, but in the former case the average velocity is somewhat smaller. No element of mass greater than phosphorus (31) has been found to yield *H*-atoms. It would appear as if the constitution of the nucleus undergoes some marked change at this stage.

It should be remarked that the disintegration observed in these experiments is on a very minute scale. Only about one  $\alpha$ -particle in a million is able to get close enough to a nucleus to effect its disintegration.

So far we have only been able to observe those fragments of atoms which escape with sufficient speed to travel further than the  $\alpha$ -particles. Another very important method of examining the effects produced within the range of the  $\alpha$ -particle has been recently examined by Mr. Shimizu. This depends on the discovery of Mr. C. T. R. Wilson that the tracks of ionising radiations can be made visible by sudden expansion of a moist gas, so that each ion becomes the centre of a visible globule of water. Wilson had previously observed an occasional sharp bend in the track of an  $\alpha$ -particle, with a short spur attached, indicating the collision of an  $\alpha$ -particle.



with an oxygen or nitrogen nucleus. By taking a large number of photographs of tracks of  $\alpha$ -particles, Mr. Shimizu found a number of cases in which the track of the  $\alpha$ -particle near the end of its range showed two nearly equal forks. It can readily be shown from the range and angle between the forks that these effects cannot be ascribed to a collision of the  $\alpha$ -particle with a  $H$ -atom, or with a nucleus of oxygen or nitrogen. It would appear not unlikely that these forks indicate an actual disruption of the atom in which a helium nucleus is released. While this conclusion is only tentative, it will be of great interest to follow up further this new method of attack of a fundamental problem. It is remarkable that while only one  $\alpha$ -particle in a million is able to liberate a  $H$ -atom from nitrogen, about one  $\alpha$ -particle in 300 appears to show a forked track, indicating that this type of disintegration occurs much more frequently than the liberation of a  $H$ -atom.

If this interpretation proves to be correct, it shows that the amount of energy required to liberate a helium nucleus from a complex nucleus of a light atom is not great. Such a result is not inconsistent with modern ideas of the relation between mass and energy, for the fact that the atomic masses of carbon and oxygen are very nearly integral multiples of the mass of the helium atom is an indication that the helium nuclei are bound loosely together. On the other hand, if we suppose the helium nucleus itself to be composed of four hydrogen nuclei and two electrons, the loss of mass in the combination indicates that the helium nucleus is so stable a structure that it should not be dissociated by even the swiftest  $\alpha$ -particle. This conclusion is supported by experimental observations as far as they have gone.



*ADDENDUM.—Fig. 7 of Paper on “A New Registering Microphotometer.” By DR. W. J. H. MOLL, University of Utrecht. (See Vol. XXXIII., Part IV., 1921, p. 207.)*

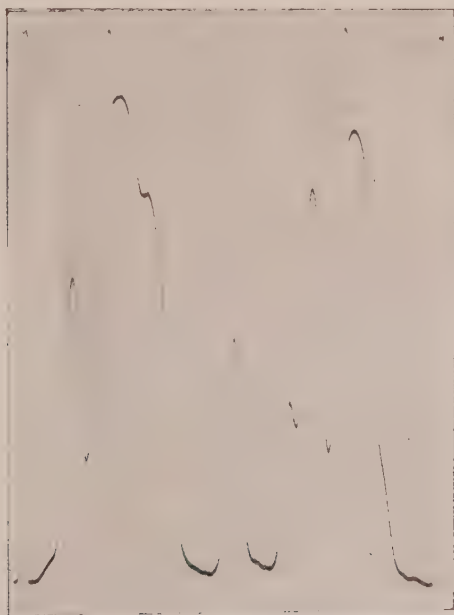


FIG. 7.—RECORD OF THE GREEN MERCURY LINE AS GIVEN BY AN ECHELON.

The three sharp points at the top have been obtained by intercepting the beam of light, by means of a screen in front of the thermopile, for one second.



# INDEX.

## A.

	Page
Absorption of Gases in Discharge Tube .....	73
Allen, N. A., <i>on</i> Carbon Arc.....	62
Analysis, Micro-, of Gases .....	287
Appleton, E.V., <i>on</i> Retroaction in Triode Oscillator .....	100
Arc, Carbon, Current Density in.....	62
Archer, R. M., <i>see</i> Owen, Dr. D.	
Atoms, Stability of .....	389

## B.

Barr, Dr. Guy, <i>on</i> Electric Relay .....	53
Bond, W. N., <i>on</i> Viscosity in Orifice Flows .....	225
Bragg, Prof., Sir W. H., <i>on</i> X-ray Reflection by Diamond .....	304
Bragg, Prof. Sir W. H., <i>on</i> X-ray Spectra of Minute Crystals ....	222
Butterworth, S., <i>on</i> Inductometers.....	312

## C.

Capacity and Eddy Current Effects in Inductometers .....	312
Chree, Dr. C., <i>on</i> an Electro-culture Problem .....	377
Clack, B. W., <i>on</i> Coefficient of Diffusion in Saturated Solutions ..	259
Crystals, Structure of Minute .....	222
Current Density in Carbon Arc .....	62

## D.

Detectors, Vacuum Tube .....	249
Diamond, Intensity of X-ray Reflection by .....	304
Diffusion, Coefficient of, in Saturated Solutions .....	259
Discussions :—	
X-ray Spectra.....	1
Resistance, Electrical, Absolute Measurement of, and Instru- ments based on Temperature-Variation of.....	125

## E.

Eddy Current Effects in Inductometers.....	312
Electrical Resistance, Absolute Measurement of, and Instruments based on Temperature-Variation of .....	125
Electro-culture Problem, An .....	377
Encounters between Non-spherical Gas Molecules .....	362

## F.

	Page
Fluospar, Reflection of X-rays from .....	297
Frequencies, Measurement of .....	243
Fringe System in Uncompensated Interferometers .....	40

## G.

Gases, Micro-analysis of .....	287
General Electric Co., Micro-analysis of Gases .....	287
Goucher, F. S., <i>on</i> Ionisation, &c., in Helium .....	13
Gravitating Fluid, Sphere of .....	116
Griffiths, Dr. A., <i>and</i> Mrs. C. H., <i>on</i> Viscosity of Water at Low Rates of Shear .....	231
Griffiths, Dr. E., <i>on</i> Specific Heat Apparatus .....	355
Guild, J., <i>on</i> Fringe Systems in Uncompensated Interferometers ..	40
Guild, J., <i>on</i> Location of Interference Fringes.....	32
Guild, J., <i>on</i> Refractometry of Prisms .....	179
Guthrie Lecture, Sixth .....	275

## H.

Hartshorn, L. ( <i>with</i> Keeping, E. S.), <i>on</i> Vacuum Tube Detectors..	249
Helium, Ionisation, &c., in .....	13
Herroun, Prof. E. F. ( <i>with</i> Wilson, Prof. E.), <i>on</i> Magnetic Susceptibility of Oxides .....	196
Hot-wire Inclinator .....	171
Humphry, R. H., <i>on</i> Hot-wire Inclinator .....	171

## I.

Inclinometer, Hot-wire .....	171
Inductometers, Capacity and Eddy Current Effects in .....	312
Interference Fringes, Location of .....	32
Interference Methods, Some Recent Applications of .....	275
Interferometers, Fringe Systems in .....	40
Ionisation, &c., by Electron Impact in Helium .....	13

## K.

Keeping, E. S., *see* Hartshorn, L.

## L.

Lees, Dr. C. H., <i>on</i> Equilibrium of Fluid Sphere .....	116
Location of Interference Fringes.....	32

## M.

	Page
Magnetic Susceptibility of Oxides .....	196
Michelson, Prof. A. A., <i>on</i> Some Recent Applications of Interference Methods (Sixth Guthrie Lecture) .....	275
Micro-analysis of Gases .....	287
Microphotometer, Registering .....	207
Molecules, Gaseous, Non-spherical, Encounters between .....	362
Moll, Dr. W. J. H., <i>on</i> a Registering Microphotometer .....	207, 395

## N.

Nagaoka, H., <i>on</i> Neon Lines and Runge's Rule .....	83
Neon Lines and Runge's Rule .....	83
Newman, F. H., <i>on</i> Absorption of Gases in Discharge Tubes .....	73
Newman, F. H., <i>on</i> Sodium Vapour Discharge Tube .....	70

## O.

Optical System, Tracing Rays through .....	174
Owen, Dr. D. ( <i>with</i> Archer, R. M.), <i>on</i> Quickness of Response in Thermionic Tube .....	104
Oxides, Magnetic Susceptibility of .....	196

## P.

Palladium, X-ray Spectrum of, Reflection from Fluorspar .....	297
Parry, H., <i>on</i> Measurement of Power by Quadrant Electrometer ..	217
Partridge, G. F., ( <i>see</i> Smith, B. S.)	
Pealing, H., <i>on</i> X-ray Reflection from Fluorspar .....	297
Pletts, J. St. V., <i>on</i> Slide Rule Improvements .....	57
Power, Measurement of, by Quadrant Electrometer .....	217
Prisms, Refractometry of .....	179

## Q.

Quadrant Electrometer, Measurement of Power by .....	217
Quickness of Response of Current to Voltage in Thermionic Tube ..	104

## [R.

Radiation, Excitation by Electron Impact in Helium .....	13
Raman, Prof. C. V., Demonstration of Optical and Acoustical Experiments .....	388
Refractometry of Prisms .....	179
Relay for Moderately Heavy Currents .....	53
Resistance, Electrical, Absolute Measurement of, and Instruments based on Temperature-Variation of .....	125
Response in Thermionic Tube, Quickness of .....	104
Retroaction in Triode Oscillator .....	100
Runge's Rule and Neon Lines .....	83
Rutherford, Prof. Sir E., <i>on</i> The Stability of Atoms .....	389



## S.

	Page
Slide Rule Improvements.....	57
Smith, B. S. ( <i>with</i> Partridge, G. F.), <i>on</i> Frequency Measurement....	243
Smith, T., <i>on</i> Tracing Rays through Optical System.....	174
Sodium Vapour Discharge Tube .....	71
Specific Heat Apparatus.....	355
Spectrometer, Ionisation, and Structure of Minute Crystals.....	222
Spectrum, ( <i>see</i> X-ray Spectra)	
Sphere of Gravitating Fluid, Equilibrium of .....	116
Stability of Atoms .....	389
Susceptibility, Magnetic, of Oxides .....	197

## T.

Temperature-Variation of Resistance, Instruments based on .....	125
Thermal Transpiration Currents.....	266
Thermionic Tubes or Valves.....	100, 104
Tracing Rays through an Optical System .....	174
Transpiration Currents, Thermal.....	266
Triode ( <i>see</i> Thermionic)	

## V.

Vacuum-Tube Detectors .....	249
Valves ( <i>see</i> Thermionic Valves)	
Viscosity, Effect of, on Orifice Flows .....	225
Viscosity of Water at Low Rates of Shear .....	231

## W.

Water, Viscosity of, at Low Rates of Shear.....	231
West, Dr. G. D., <i>on</i> Thermal Transpiration Currents .....	267
Wilson, Prof. E. ( <i>see</i> Herroun, Prof. E. F.)	

## X.

X-rays, Intensity of Reflection of, by Diamond .....	304
X-ray Spectra, Discussion on .....	1
X-ray Spectra of Minute Crystals .....	223
X-ray Spectrum of Palladium, Reflection from Fluorspar of .....	297

# **PUBLICATIONS OF THE PHYSICAL SOCIETY.**

## **THE SCIENTIFIC PAPERS**

OF THE LATE

**SIR CHARLES WHEATSTONE, F.R.S.**

*Demy 8vo. cloth. Price 12s.; to Fellows, 6s.*

*Uniform with the above.*

## **THE SCIENTIFIC PAPERS OF**

**JAMES PRESCOTT JOULE, D.C.L., F.R.S.**

Vol. I. 4 Plates and Portrait, price 18s.; to Fellows, 9s.

Vol. II. 3 Plates, price 12s.; to Fellows, 6s.

## **PHYSICAL MEMOIRS.**

**PART I.—VON HELMHOLTZ**, On the Chemical Relations of Electrical Currents. Pp. 110. *Price 6s.; to Fellows, 3s.*

**PART II.—HITTOFF**, On the Conduction of Electricity in Gases; **PULUJ**, Radiant Electrode Matter. Pp. 222. *Price 12s.; Fellows 6s.*

**PART III.—VAN DER WAALS**, On the Continuity of the Liquid and Gaseous States of Matter. Pp. 164. *Price 12s.; to Fellows, 6s.*

## **REPORT ON RADIATION AND THE QUANTUM-THEORY.**

By **J. H. JEANS, M.A., F.R.S.**

*Price 9s.; to Fellows, 4s. 6d. Bound in Cloth, 12s. 9d.; to Fellows, 8s. 3d.*

## **REPORT ON THE RELATIVITY THEORY OF GRAVITATION.**

By **A. S. EDDINGTON, M.A., M.Sc., F.R.S.**

*Plumian Professor of Astronomy and Experimental Philosophy, Cambridge.*

**Third Edition.**—*Price 6s.; to Fellows 3s. Bound in cloth, 8s. 6d., to Fellows 6s.*

## **THE TEACHING OF PHYSICS IN SCHOOLS.**

*Price to Non-Fellows, 1s. 6d. net, post free 1s. 8d.*

## **METROLOGY IN THE INDUSTRIES.**

*Price to Non-Fellows, 1s. 6d. net, post free 1s. 8d.*

## **DISCUSSION ON LUBRICATION.**

*Price to Non-Fellows, 1s. 6d. net, post free 1s. 8d.*

## **PROCEEDINGS.**

The "Proceedings" of the Physical Society can be obtained at the following prices:—

Vol. I. (3 parts) bound cloth, 22s. 6d.

Vols. II., IV., V., XXIII., XXV., XXVI., XXVII., XXVIII., XXIX., XXX. & XXXI. (5 parts each), cloth, 34s. 6d.

Vols. III., VI. to XII. & XXII. (4 parts each), bound cloth, 28s. 6d.

Vol. XIII. (13 parts, each containing Abstracts), bound cloth (without Abstracts), 70s. 6d.

Vols. XIV. & XV. (12 parts, each containing Abstracts), bound cloth (without Abstracts), 34s. 6d.

Vols. XVI. & XIX. (8 parts each), bound cloth, 52s. 6d.

Vols. XVII., XVIII. & XXI. (7 parts each), bound cloth, 46s. 6d.

Vols. XX. & XXIV. (6 parts), bound cloth, 40s. 6d.

Most of the parts can be purchased separately, price 6s., by post 6s. 3d. Fellows can obtain the *Proceedings* (in parts) for their personal use at half the above prices.

## **ABSTRACTS OF PHYSICAL PAPERS FROM FOREIGN SOURCES.**

Vols. I. (1895), II. (1896), III. (1897), 22s. 6d. each; Fellows, 11s. 3d.

*Strong cloth cases for binding the "Proceedings," price 3s. 6d. each, post free.*

**BLAKESLEY, T. H.** A Table of Hyperbolic Sines and Cosines. *Price 2s. 3d.; to Fellows, 1s. 2d.*

**LEHFELDT, R. A.** A List of Chief Memoirs on the Physics of Matter. *Price 3s.; to Fellows, 1s. 6d.*

*Applications for the above Publications should be sent direct to*

**FLEETWAY PRESS, LTD.**

3-9, DANE STREET, HIGH HOLBORN, LONDON, W.C.1.



## CONTENTS.

	PAGE
XXV. A Method for the Micro-Analysis of Gases. From the Research Laboratory of the GENERAL ELECTRIC Co. . .	287
XXVI. On the Reflection of the X-Ray Spectrum of Palladium from Fluorspar. By H. PEALING, M.Sc. . . . .	297
XXVII. The Intensity of the X-Ray Reflection by Diamond. By Prof. Sir W. H. BRAGG, K.B.E., F.R.S. . .	304
XXVIII. Capacity and Eddy Current Effects in Inductometers. By S. BUTTERWORTH, M.Sc. . . . .	312
XXIX. New Specific Heat Apparatus. By EZER GRIFFITHS, D.Sc. . . . .	355
XXX. On Encounters between Non-Spherical Gas Molecules. By Prof. A. O. RANKINE, D.Sc. . . . .	362
XXXI. An Electro-Culture Problem. By C. CHREE, Sc.D., F.R.S. . . . .	377
Demonstration of Acoustical and Optical Experiments. By Prof. C. V. RAMAN. . . . .	388
The Stability of Atoms. Lecture by Sir ERNEST RUTHERFORD, F.R.S. . . . .	389
ADDENDUM.—Fig. 7 of Paper on A New Registering Microphotometer. By Dr. W. J. H. MOLL, University of Utrecht. ( <i>See Vol. XXXIII., Part IV., 1921, p. 207</i> ) ..	395